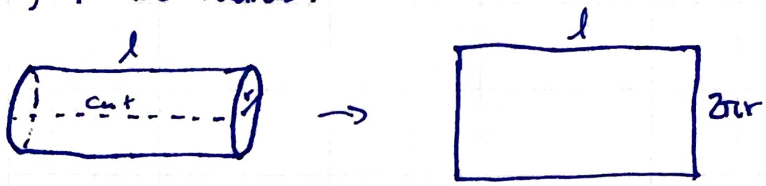
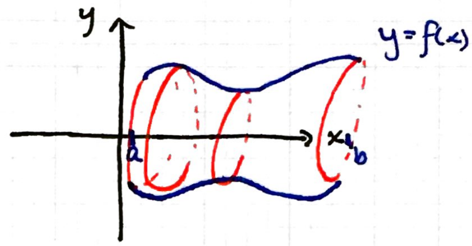
8.2 Area of Surface of Revolution

<u>Motivation</u> From elementary geometry, we know the lateral surface area of a right circular cylinder is $2\pi r l$, where l is the length of the cylinder, r the radius.



Our goal is to calculate the lateral surface area of more general surfaces of revolution



In revolving about the x axis, the radius becomes $y = f(x)$, and the length is the length of the arc between a and b. After some of the same tricks we've seen before, we arrive at the formula

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx}$$

If the curve is described as $x = g(y)$ for $c \le x \le d$, then the formula becomes

$$\boxed{S = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2}\, dy}$$

We can write more succinctly for a rotation about the x-axis

$$\boxed{S = \int 2\pi y\, ds}$$

and about the y-axis

$$\boxed{S = \int 2\pi x\, ds}$$

where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$.

Ex 1 Find area of surface obtained by rotating $y = \sqrt{5-x}$ about x-axis, $3 \le x \le 5$.

Solution: $\frac{dy}{dx} = \frac{-1}{2\sqrt{5-x}}$.

So $$S = \int_3^5 2\pi \sqrt{5-x} \cdot \sqrt{1 + \frac{1}{4(5-x)}}\, dx$$

$$\cancel{= \int_3^5 2\pi \sqrt{5-x} \sqrt{\frac{21-4x}{20-4x}}\, dx}$$

$u = 5-x$
$du = -dx$

$$= -\int_2^0 2\pi\, u^{1/2} \sqrt{1 + \frac{1}{4u}}\, du$$

$$= 2\pi \int_0^2 \sqrt{u + \frac{1}{4}}\, du$$

$$= 2\pi \int_0^2 \sqrt{\frac{4u+1}{4}}\, du$$

$$= \pi \int_0^2 \sqrt{4u+1}\, du$$

$s = 4u+1$
$ds = 4ds$

$$= \frac{\pi}{4} \int_1^9 s^{1/2}\, ds$$

$$= \boxed{\frac{13\pi}{3}}$$

Ex2 $y = \cos\left(\frac{1}{2}x\right), \quad 0 \le x \le \pi$

$$\frac{dy}{dx} = -\frac{1}{2}\sin\left(\frac{1}{2}x\right) \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\sin^2\left(\frac{1}{2}x\right)$$

$$S = 2\pi \int_0^{\pi} \cos\left(\frac{x}{2}\right)\sqrt{1+\frac{1}{4}\sin^2\left(\frac{x}{2}\right)}\, dx \qquad u = \frac{1}{2}\sin\left(\frac{x}{2}\right), \quad du = \frac{1}{4}\cos\left(\frac{x}{2}\right)dx$$

$$= 8\pi \int_0^{1/2} \sqrt{1+u^2}\, du$$

(#21) $$= 8\pi\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right)\Big|_0^{1/2}$$

$$= 8\pi\left[\left(\frac{1}{4}\cdot\frac{\sqrt{5}}{2} + \frac{1}{2}\ln\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)\right)\right.$$

Ex3 Area of $x^{2/3} + y^{2/3} = 1, \quad 0 \le y \le 1$

$$0 \le y \le 1 \Rightarrow 0 \le x \le 1$$

$$y^{2/3} = 1 - x^{2/3}$$

$$y = (1-x^{2/3})^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}(1-x^{2/3})^{1/2} \cdot -\frac{3}{2}x^{-1/3}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = x^{-2/3}(1-x^{2/3}) = x^{-2/3} - 1$$

So $$S = \int_0^1 2\pi x \sqrt{1+(x^{-2/3}-1)}\, dx$$

$$= \int_0^1 2\pi x \sqrt{x^{-2/3}}\, dx$$

$$= \int_0^1 2\pi x^{2/3}\, dx$$

$$= \frac{6\pi}{5}$$

Ex 4 Gabriel's Horn.

Rotate $y = 1/x$ about x-axis for $1 \le x < \infty$.

Then from chap 7, we know

$$V = \int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \lim_{t\to\infty} -\pi \frac{1}{x}\Big|_1^t = \lim_{t\to\infty}\left(-\frac{\pi}{t} + \frac{\pi}{1}\right) = \pi$$

But Surface Area

$$y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{-1}{x^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^4}$$

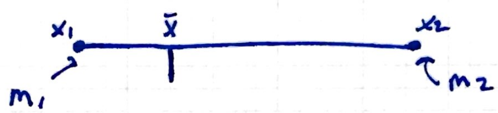
So $$S = \int_1^\infty 2\pi \cdot \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx$$

$$> \int_1^\infty 2\pi \frac{1}{x}\, dx = \infty$$

8.3 Moments and Center of mass

Center of mass point where object balances horizontally.

If we work with the x-axis,



Then $$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}.$$

The numbers $x_1 m_1$ and $x_2 m_2$ are the moments of m_1 and m_2 wrt the origin.

For a general discrete system,

$$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

$M = \Sigma m_i x_i$ is called the moment of the system about the origin.

In the plane, we define

$M_y = \Sigma m_i x_i$ - the moment about the y-axis

$M_x = \Sigma m_i y_i$ - the moment about the x-axis

Measures the tendency to rotate about respective axis

If $m = \sum m_i$, then

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}.$$

For a flat plate with unform density ρ,

$$M_y = \rho \int_a^b x\, f(x)\, dx$$

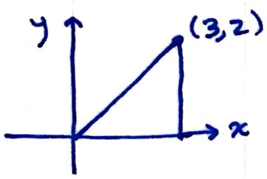
$$M_x = \rho \frac{1}{2}\int_a^b \frac{1}{2}[f(x)]^2\, dx$$

and $m = \rho A = \rho \int_a^b f(x)\, dx$

So

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx, \quad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\, dx$$

Ex 5 Calculate M_x, M_y and $(\bar{x}, \bar{y})$ for



with $\rho = 4$

$A = \frac{1}{2}(3)(2) = 3$, so $m = \rho A = 4 \cdot 3 = 12$

$$M_x = \rho \int_0^3 \frac{1}{2}\left(\tfrac{2}{3}x\right)^2 dx = 2 \cdot \frac{4}{9}\int_0^3 x^2\, dx$$

$$= \frac{8}{9} \cdot \frac{1}{3} x^3 \Big|_0^3 = \boxed{8}$$

$$M_y = \rho \int_0^3 x\left(\tfrac{2}{3}x\right) dx = \frac{8}{3}\int_0^3 x^2\, dx = \frac{8}{9} x^3 \Big|_0^3 = 24$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{24}{12}, \frac{8}{12}\right) = \left(2, \frac{2}{3}\right)$$