A <u>sequence</u> is just a list of numbers in a particular order.

$a_1, a_2, a_3, \dots, a_n$

We can also think of a sequence as given by a function defined on the natural numbers with values $a_n$ in $\mathbb{R}$.

We also denote sequences by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Sometimes given a sequence we want to find the general $a_n$ term.

<u>Ex 1</u>

$4, -1, \frac{1}{4}, \frac{-1}{16}, \frac{1}{64}, \dots,$

$4^1 \quad (-1)(4^0) \quad (-1)^2(4)^{-1}, \dots \quad = \frac{(-1)^n 4}{4^n} = \frac{(-1)^n}{4^{n-1}} = a_n$

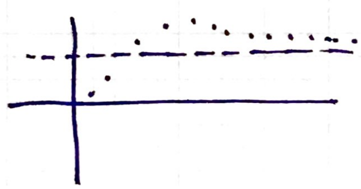
$n=0$

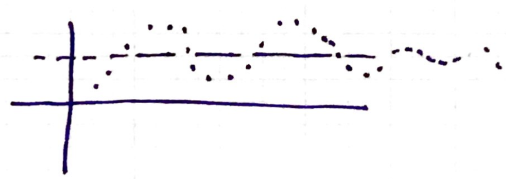
<u>Def</u> A sequence $\{a_n\}$ has the <u>limit</u> $L$, and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms $a_n$ close to $L$ for $n$ large.
If $\lim_{n\to\infty} a_n$ exists, we say the sequence <u>converges</u>. Otherwise, we say the sequence <u>diverges</u>. (See $\varepsilon$ def)

<u>Ex</u>





<u>Theorem</u> If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when $n$ is an integer then

$$\lim_{n\to\infty} a_n = L$$

<u>Limit laws for sequences</u> If $\{a_n\}, \{b_n\}$ converge, $c \in \mathbb{R}$,

$\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$

$\lim (c a_n) = c \cdot \lim a_n, \quad \lim c = c$

$\lim (a_n b_n) = (\lim a_n)(\lim b_n)$

$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$ if $\lim b_n \neq 0$

$\lim a_n^p = (\lim a_n)^p$ if $p>0$ and $a_n>0$

<u>Squeeze Theorem</u>

If $a_n \le b_n \le c_n$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

<u>Theorem</u> If $\lim |a_n| = 0$, then $\lim a_n = 0$.

<u>Ex 2</u>

$$\lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2(3/n^2 + 5)}{n^2(1/n + 1)} = \lim_{n\to\infty} \frac{3/n^2 + 5}{1/n + 1} = 5$$

<u>Ex3</u>

$$\lim \frac{n}{\sqrt{10+n}} = \lim \frac{1}{\frac{1}{n}\sqrt{10+n}} = \lim \frac{1}{\sqrt{10/n^2 + 1/n}} = \infty$$

<u>Ex4</u> $(-1)^n/n$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim \frac{1}{n} = 0 \Rightarrow \lim (-1)^n/n \to 0.$$

<u>Theorem</u> If $\lim a_n = L$ and the function $f$ is continuous at $L$, then

$$\lim f(a_n) = f(L).$$

<u>Ex5</u> $\lim_{n\to\infty} \cos\left(\frac{n\pi}{n+1}\right)$

$\frac{n\pi}{n+1} \to \pi$ as $n \to \infty$ and $\cos\pi = -1$ and cosine cts at $\pi$

$\Rightarrow \lim = -1$.

<u>Ex6</u> $\frac{n!}{n^n}$

$$\frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot \cdots \cdot n}{n\ n\ n\ \cdots\ n} = a_n = \frac{1}{n}\Big(\overbrace{\frac{2\cdot 3\cdots n}{n\ n\cdots n}}^{\le 1}\Big)$$

$\Rightarrow \quad 0 < a_n \le \frac{1}{n}$

Squeeze Thm $\Rightarrow$ $\lim = 0$.

$$\lim r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \end{cases}$$

Def A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ $\forall n \geq 1$, decreasing if $a_n > a_{n+1}$ $\forall n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.

Two ways to test $\nearrow$ or $\searrow$

Ex

$$a_n = \frac{3}{n+4}$$

Then $\frac{3}{n+4} > \frac{3}{(n+1)+4} = \frac{3}{n+5}$ $\Rightarrow$ decreasing

Ex $\frac{n}{n^2+1}$

Consider $f(x) = \frac{x}{x^2+1}$. Then $f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ if $x^2 > 1$

$\Rightarrow$ Sequence is decreasing for $n > 1$.

Def A sequence is bounded above if there is a number $M$ such that $a_n \leq M$ for all $n \geq 1$.

Bounded below if there is $m$ st

$m \leq a_n$ for all $n \geq 1$.

Monotonic sequence Thm Every bounded monotonic sequence is convergent.