11.2 Series

A <u>series</u> is what we get if we add all the terms in an infinite sequence

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \sum a_n$$

$\uparrow$ if the indices are understood from the context

<u>Ex 1</u> $1+2+3+\cdots = \sum_{n=1}^{\infty} n$

<u>Ex 2</u> $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

<u>Def</u> the <u>nth partial sum</u> of a series is the sum of the first n terms

<u>Ex</u> $1+2+3+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

<u>Ex</u> $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \sum_{j=1}^{n} \frac{1}{2^n} = 1 - 2^{-n}$

<u>Def</u> Let s_n denote the nth partial sum of the series $\sum a_n$. We say the series $\sum a_n$ is <u>convergent</u> if $\lim_{n\to\infty} s_n = s$ ($\{s_n\}$ is convergent as a sequence). s is called the <u>sum</u> of the series. If $\{s_n\}$ diverges, we say the series $\sum a_n$ is <u>divergent</u>.

<u>Ex</u> Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=0}^{\infty} ar^n$$

If $r=1$, then $\{s_n\} = \{na_n\} \to \infty$. If $r \neq 1$

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ -\; rs_n &= -(\quad ar + ar^2 + \cdots + ar^{n-1} + ar^n) \end{aligned}$$

$$s_n - rs_n = a - ar^n$$

$$s_n(1-r) = a(1-r^n)$$

$$s_n = \frac{a(1-r^n)}{1-r} \to \frac{a}{1-r} \quad \text{if } |r|<1$$

$$\to \infty \quad \text{if } |r| \geq 1$$

Ex $.9999\cdots$

$$= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots = \sum_{n=1}^{\infty} \frac{9}{10^n}$$

$$= \frac{9}{10} \sum_{n=1}^{\infty} \frac{1}{10^{n-1}} = \frac{9}{10} \sum_{n=0}^{\infty} \frac{1}{10^n}$$

$$= \frac{9}{10}\left(\frac{1}{(1-\frac{1}{10})}\right) = \frac{9}{10}\left(\frac{1}{9/10}\right) = 1$$

~~Ex~~ Remark We can say that the sum of a (convergent) geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

Ex $\frac{3}{16} + \frac{3}{32} + \frac{3}{64} + \cdots$

$$= 3\left(\frac{1}{16} + \frac{1}{32} + \frac{1}{64}\right) = 3\left(\frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots\right)$$

$$= \frac{3}{2^4}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = \frac{3}{2^4} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{3}{2^4}\left(\frac{1}{1-\frac{1}{2}}\right) = \boxed{\frac{3}{2^3}} = \frac{3}{8}$$

Or First term: $\frac{3}{16}$

Common ratio: $1/2$

$$S = \frac{3/16}{1-\frac{1}{2}} = 2 \cdot \frac{3}{16} = \frac{3}{8}$$

Algebraically, this means $\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$

Ex $\sum_{n=1}^{\infty} 3^{n+1} 4^{-n}$

$$= \sum_{n=1}^{\infty} 3\left(\frac{3}{4}\right)^n = \frac{3 \cdot 3/4}{1 - 3/4} = 4 \cdot \frac{9}{4} = 9$$

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof We consider the partial sums $s_2, s_4, s_8, \ldots$

$$s_2 = 1 + \tfrac{1}{2}$$
$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2}$$
$$s_8 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right)$$
$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right)$$
$$= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}$$

Keep going... $s_{16} > 1 + \frac{4}{2}$, $s_{32} > 1 + \frac{5}{2}, \ldots$

So $s_{2^n} > 1 + \frac{n}{2} \implies s_{2^n} \to \infty$ as $n \to \infty \implies \{s_{2^n}\}$ is divergent $\implies \{s_n\}$ is divergent. Thus $\sum 1/n$ diverges. ∎

Theorem If the series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. By definition, $\{s_n\}$ is convergent, so $\lim_{n\to\infty} s_n = s$. But $n-1 \to \infty$ as $n \to \infty$ so $\lim_{n\to\infty} s_{n-1} = s$ as well. So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Careful The converse is not true. That is, $\lim a_n = 0$ does not imply $\sum a_n < \infty$. The harmonic series provides a counter example.

Test for divergence If $\lim a_n$ dne or $\lim a_n \neq 0$, then $\sum a_n$ is divergent. (Follows from the theorem.)

Theorem If $\sum a_n$, $\sum b_n$ are convergent then so are $\sum c a_n$, $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$, and

a) $\sum c a_n = c \sum a_n$

b) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

c) $\sum a_n - b_n = \sum a_n - \sum b_n$

<u>Ex</u> Find $s = \sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$

<u>Solution</u>
$$= \sum 3 \cdot \frac{1}{n(n+1)} + \sum \frac{1}{2^n}$$
$$= 3 \sum \frac{1}{n(n+1)} + \underbrace{\sum \frac{1}{2^n}}_{=1}$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$
$$1 = A(n+1) + Bn$$

$n=-1$: $1 = -B \Rightarrow B = -1$
$n=0$: $1 = A$

So $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

$$s_m = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) + \left(\frac{1}{m} - \frac{1}{m+1}\right)$$
$$= 1 - \frac{1}{m+1} \to 1$$

So $3 \sum \frac{1}{n(n+1)} = 3 \cdot 1$, and $s = 3 \cdot 1 + 1 = \boxed{4}$

<u>Remark</u> $\sum \frac{1}{n} - \frac{1}{n+1}$ is called a <u>telescoping sum</u>

<u>Remark</u> Convergence of a series depends only on the tail end of the sum. If we know $\sum_{n=N+1}^{\infty} a_n$ converges, then
$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{N} a_n}_{\text{finite}} + \underbrace{\sum_{n=N+1}^{\infty} a_n}_{\text{finite (since it converges)}}$$

So $\sum_{n=1}^{\infty} a_n$ converges

Ex Find the values of x for which the series

$$\sum_{n=0}^{\infty} (-4)^n (x-5)^n$$

converges. Find the sum of the series for those values of x.

Solution This is a geometric series.

$$\sum (-4)^n (x-5)^n = \sum \left(\frac{x-5}{-4}\right)^n, \quad \text{so} \quad r = \frac{x-5}{-4},$$

so this converges if and only if

$$|r| = \left|\frac{x-5}{-4}\right| = \left|\frac{x-5}{4}\right| < 1$$

$$\Leftrightarrow |x-5| < 4$$

$$\Leftrightarrow -4 < x-5 < 4$$

$$\Leftrightarrow 1 < x < 9$$

And the sum is

$$\frac{1}{1-\left(\frac{x-5}{-4}\right)} = \frac{1}{1+\frac{x-5}{4}} = \frac{1}{\frac{4+x-5}{4}}$$

$$= \boxed{\frac{4}{x-1}}$$