<u>The comparison test</u> Suppose $\sum a_n$, $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum a_n$ is also divergent.

<u>Ex 1</u> $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ is convergent.

<u>proof</u>

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2} \Rightarrow \sum \frac{5}{2n^2+4n+3} < \sum \frac{5}{2n^2}$$

And $\sum \frac{5}{2n^2} = \frac{5}{2} \sum \frac{1}{n^2}$, which is a convergent p-series, so the original series converges by Comparison test.

<u>Remark</u> We usually choose a p-series or geometric series for our $\sum b_n$.

<u>Remark</u> The inequality in the comparison test need only hold for all $n \ge N$ for some $N$.

<u>Ex 2</u> The series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

<u>proof</u> Since $\ln e = 1$ and $e \sim 2.718$, we know for $n \ge 3$ $\ln(n) > 1$. So $\dfrac{\ln(n)}{n} > \dfrac{1}{n}$ for all $n \ge 3$. But $\sum 1/n$ diverges, so the series diverges by comparison test.

**Remark** Having the correct inequality is essential

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

$\frac{1}{2^n - 1} > \frac{1}{2^n}$, so we cannot apply the CT with the series $\sum \frac{1}{2^n}$.

**The Limit Comparison Test** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $0 < c < \infty$, ~~then~~ then either both series converge or both series diverge.

**Ex3** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

*proof* Consider $b_n = \frac{1}{2^n}$. Then

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0.$$

So the series converges by LCT since $\sum \frac{1}{2^n}$ is a convergent geom series with $r = \frac{1}{2} < 1$.

Ex 4 $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converges.

proof Look at largest power of $n$ in numerator and denominator:

$$b_n = \frac{2n^2}{n^{5/2}}$$

Then $\lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{2n^2/n^{5/2}} = \lim_{n\to\infty} \frac{2n^2+3n}{(5+n^5)^{1/2}} \cdot \frac{n^{5/2}}{2n^2}$

$= \lim_{n\to\infty} \frac{2n^2+3n}{2(5+n^5)^{1/2}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2(5+n^5)^{1/2}}$

$= \lim_{n\to\infty} \frac{2+3/n}{2(5/n^5+1)^{1/2}} = 1.$

Since $\sum b_n$ is a convergent p-series, the original series converges by LCT.

Practice

1) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

2) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$