An <u>alternating series</u> is a series whose terms alternate signs.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

We can think of an alternating series in two parts

$$\sum (-1)^n b_n,$$

where b_n is a sequence of positive terms.

<u>Alternating Series Test</u> If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n, \quad (b_n > 0 \text{ for all } n)$$

satisfies
(i) $b_{n+1} \leq b_n$ for all n
(ii) $\lim_{n\to\infty} b_n = 0$

then the series is <u>convergent</u>.

<u>Ex 1</u> The alternating harmonic series is convergent.

$\frac{1}{n} \to 0$ and is clearly decreasing $\Rightarrow$ convergent by AST.

<u>Ex 2</u> $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

It's clear that $b_n \to 0$. Decreasing? Take d/dx!

$$\frac{d}{dx}\left(\frac{x^2}{x^3+4}\right) = \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2}$$

$$= \frac{2x^4 + 8x - 3x^4}{(x^3+4)^2}$$

$$= \frac{x(-x^3+8)}{(x^3+4)^2}$$

And $-x^3+8 = 0$ when $x=2$. Easy to see decreasing on $(-2,\infty)$. So $\frac{n^2}{n^3+4}$ is decreasing for $n > 2$. For $n=1, 2$, not the case, but we really just need <u>eventually</u> decreasing. Now it follows that the series converges by AST.

Ex3 $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1 \Rightarrow$ divergent by test for divergence.

Important It's necessary to check whether the sequence $\{b_n\}$ actually decreases to 0.

Ex $\frac{1}{1^2} - \frac{1}{1} + \frac{1}{2^2} - \frac{1}{2} + \frac{1}{3^2} - \frac{1}{3} + \cdots$

We can write this in Σ notation as

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n}\right).$$

~~If this series can~~

But if we look at the partial sums, after some work, we would find out this series diverges.

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$ is the sum of an alternating series that satisfies

i) $b_{n+1} \le b_n$ and ii) $\lim_{n\to\infty} b_n = 0$

$$|R_n| = |s - s_n| \le b_{n+1}$$

Ex 4 How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 2^n}$ do we need to add in order to have error $< .0005$?

$$b_{n+1} = \frac{1}{(n+1)^2 2^{(n+1)}} \underset{\text{want}}{<} \frac{1}{2000} \Rightarrow 2000 < (n+1)^2 2^{(n+1)}$$

Try $n=4$: $5^2 \cdot 2^5 = 800$ ✗

$n=5$: $6^2 \cdot 2^6 = 2304$ ✓

So we need to add 5 terms to get the desired error.

Ex5 Estimate the sum correct to 4 decimal places.

$$\sum_{n=1}^{\infty} \frac{(-.8)^n}{n!}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(.8)^n}{n!}$$

Want $b_{n+1} = \frac{(.8)^{n+1}}{(n+1)!} < .00009$

when $n = 5$ $\quad \frac{.8^{n+1}}{(n+1)!} = .0003$

$n=6$: $\quad \frac{(.8)^{n+1}}{(n+1)!} = .00004 < .00009$

So

$$\sum_{n=1}^{6} \frac{(-.8)^n}{n!} \approx -.5506$$ is correct to 4 decimals.