11.6 Absolute Convergence Ratio, Root test.

A series Σa_n is said to be absolutely convergent if the series $\Sigma |a_n|$ converges.

Ex1 The series $\Sigma (-1)^n \frac{1}{n^2}$ is absolutely convergent because

$\Sigma |(-1)^n \frac{1}{n^2}| = \Sigma \frac{1}{n^2}$, which is a convergent p-series.

Ex2 The alternating harmonic is not absolutely convergent because $\Sigma |(-1)^n \frac{1}{n}| = \Sigma \frac{1}{n}$, which diverges.

Def A series Σa_n is said to be conditionally convergent, if Σa_n converges, but not absolutely.

Theorem If a series Σa_n is absolutely convergent, then it is Convergent.

proof $0 \le a_n + |a_n| \le 2|a_n|$. Thus by Comparison Test, the series $\Sigma (a_n + |a_n|)$ is also convergent. So

$$\Sigma (a_n + |a_n|) - \Sigma |a_n| = \Sigma a_n$$

is convergent as well. ☒

Ex2 The series $\Sigma \frac{\cos n}{n^2}$ is convergent.

Solution $$\Sigma \left|\frac{\cos n}{n^2}\right| = \Sigma \frac{|\cos n|}{n^2}$$

and $|\cos n| \le 1 \Rightarrow \left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$ so that by CT,

$\Sigma \left|\frac{\cos n}{n^2}\right|$ converges $\Rightarrow \Sigma \frac{\cos n}{n^2}$ converges.

The Ratio Test Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

i) If $L < 1$, then the series Σa_n is absolutely convergent.

ii) If $L > 1$ or $L = \infty$, then the series Σa_n is divergent.

iii) If $L = 1$, the test is inconclusive; we don't know whether Σa_n converges or diverges.

Ex3 The series $\sum \frac{n^n}{n!}$ is divergent.

Solution

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)\,n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \longrightarrow e > 1 \quad \text{as} \quad n \to \infty$$

So Ratio Test $\Rightarrow$ the series diverges.

The Root Test Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

i) If $L < 1$, then the series $\sum a_n$ is absolutely convergent.
ii) If $L > 1$ or $L = \infty$, then the series is divergent.
iii) If $L = 1$, then the Root Test is inconclusive.

Ex4 The series $\sum \left(\frac{2n+3}{3n+2}\right)^n$

$$a_n = \frac{2n+3}{3n+2} \Rightarrow \sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2 + 3/n}{3 + 2/n} \longrightarrow \frac{2}{3} < 1$$

$\Rightarrow$ the series converges by the Root Test.

Practice

1) $\sum \frac{\sin n}{2^n}$ AC, CC, D
2) $\sum \frac{\cos(n\pi/3)}{n!}$
3) $\sum \frac{(2n)!}{(n!)^2}$
4) $\sum \left(\frac{n^2+1}{2n^2+1}\right)^n$
5 $\sum (\arctan n)^n$

1) $\left|\frac{\sin n}{2^n}\right| \le \frac{1}{2^n} \Rightarrow AC$

2) $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cos((n+1)\pi/3)}{(n+1)!} \cdot \frac{n!}{\cos(n\pi/3)}\right| = \left|\frac{\cos[(n+1)\pi/3]}{\cos(n\pi/3)} \cdot \frac{1}{n+1}\right|$$

$$\le \frac{2}{n+1} \longrightarrow 0 \Rightarrow \text{Convergent}$$

3) $\sum \frac{(2n)!}{(n!)^2}$

$$\frac{a_{n+1}}{a_n} = \frac{[2(n+1)]!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{\overset{(2n+2)(2n+1)}{\cancel{2(n+1)}}(2n)!}{(n+1)^2(n!)^2} \cdot \frac{(n!)^2}{(2n)!} = \frac{2(2n+1)}{n+1}$$

$$= \frac{2(2+1/n)}{1+1/n} \rightarrow 4 > 1 \rightarrow \underline{\text{DIVERGES}}$$

4) $\sqrt[n]{|a_n|} = \frac{n^2+1}{2n^2+1} \rightarrow \frac{1}{2} < 1 \Rightarrow$ <u>converges</u>

5) $\sqrt[n]{|a_n|} = \text{arc tan } n \longrightarrow \frac{\pi}{2} > 1 \rightarrow$ <u>diverges</u>