11.8 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

where x is a variable and the c_n's are constants called coefficients.

Basically, a polynomial with infinite terms. The sum of a power series is a function

$$f(x) = \sum c_n x^n$$

whose domain is the set of all x for which the series converges.

Ex 1 We know from 11.2 that when $c_n = 1$ for all n, the series $\sum_{n=0}^{\infty} x^n$ converges when $|x|<1$ and diverges when $|x| \geq 1$.

More generally, a power series of the form $\sum c_n (x-a)^n$ is called the power series centered at a or power series about a.

By convention 0^0 is 1 so that $x=a$ gives a convergent power series.

Ex 2 For what values of x is the series $\sum_{n=0}^{\infty} n!\, x^n$ convergent?

Solution Using ratio test:

$$\left| \frac{(n+1)!}{n!} \frac{x^{n+1}}{x^n} \right| = (n+1)|x| \to \infty \text{ as } n \to \infty$$

$\Rightarrow$ converges only for $x=0$.

Theorem For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities:

i) The series converges only when $x=a$.
ii) The series converges for all x.
iii) There is a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$.

The number R is called the radius of convergence. In case (i), $R=0$ and case (ii) $R=\infty$. The interval of convergence of the power series is the set of all x for which the series converges. In case (i), the interval is just the point a, in (ii), the interval is $(-\infty, \infty)$. In case (iii) we know the series converges for $|x-a|<R$; that is $a-R<x<a+R$. But at the end points, any thing could happen, so they must be checked separately.

Ex3 Find radius and interval of convergence.

$$\sum_{n=0}^{\infty} \underbrace{\frac{(-3)^n x^n}{\sqrt{n+1}}}_{a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left| -3x \sqrt{\frac{n+1}{n+2}} \right|$$

$$= 3\sqrt{\frac{1+1/n}{1+2/n}}\,|x| \to 3|x| \text{ as } n \to \infty$$

Ratio Test $\Rightarrow$ this converges if $3|x|<1$ and diverges if $3|x|>1$. That is, converges when $|x|<1/3$, so $x \in (-1/3, 1/3)$. Need to check convergence for $\pm 1/3$.

When $x = \frac{1}{3}$, we get

$$\sum_{n=0}^{\infty} \frac{(-3)^n (\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}},$$ which converges by AST.

But when $x = -1/3$,

$$\sum_{n=0}^{\infty} \frac{(-3)^n (-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}},$$

which is a divergent p-series with $p=\frac{1}{2}$.

So the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

Ex4 $\sum_{n=1}^{\infty} \frac{(x-a)^n}{n^n}$

$$\sqrt[n]{|a_n|} = \left|\frac{x-a}{n}\right| < 1 \quad \boxed{\text{or } \left|\frac{x-a}{n}\right| \to 0}$$

So $|x-a| < n \to |x-a| < \infty$ as $n \to \infty$

So $R=\infty$, I of C: $(-\infty, \infty)$.

Ex $\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \cdot \ln n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)} \cdot \frac{2^n \ln n}{(x+2)^n}\right| = \left|\frac{x+2}{2}\right| \cdot \frac{\ln n}{\ln(n+1)}$$

$$\lim_{n\to\infty} \frac{\log n}{\log(n+1)} = \lim_{n\to\infty} \frac{\log n}{\log[(n)(1+\frac{1}{n})]} = \lim_{n\to\infty} \frac{\log n}{\log n + \log(1+\frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{\log n}{\log n} = 1 \quad (\text{Since } \log(1+\tfrac{1}{n}) \to \log 1 = 0)$$

So we need $\left|\frac{x+2}{2}\right| < 1 \Leftrightarrow |x+2| < 2 \Leftrightarrow -2 < x+2 < 2$

$\Leftrightarrow -4 < x < 0.$

When $x=0$, get $\sum_{n=2}^{\infty} \frac{2^n}{2^n \log n} = \sum_{n=2}^{\infty} \frac{1}{\log n} = \infty$

When $x=-4$, get

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \log n} = \sum \frac{(-1)^n}{\log n} < \infty \text{ by AST.}$$

So Interval of Conv: $[-4, 2)$. R of C $= 2$