11.9 Representations of functions as power series

Already know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Can use this for other similar functions

Ex 1

$$\frac{1}{x+2} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2}\frac{1}{1-(-x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

Converges when $\left|\frac{-x}{2}\right| < 1 \iff |x| < 2$.

Ex2

$$\frac{x^3}{x+2} = x^3\frac{1}{x+2} = x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$

Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R>0$, then the function defined by

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

ii) $\int f(x)\,dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots$
$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

And the radius of convergence for both (i) and (ii) is R.

Ex3

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\frac{d}{dx}(x^n)$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

Ex4 $\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \,dx$

$= \sum_{n=0}^{\infty} \int (-1)^n x^n \,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C$

To find C: plug in $x=0$

$\ln(1+0) = \ln 1 = 0 = C$

$\Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Ex 5 $f(x) = \frac{2x+3}{x^2+5x+6} = \frac{2x+3}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$2x+3 = A(x+3) + B(x+2)$

$2(-3)+3 = B(-3+2)$ $\qquad$ $2(-2)+3 = A(-2+3)$

$-3 = B(-1)$ $\qquad$ $\boxed{-1 = A}$

$\boxed{3 = B}$

So $f(x) = \frac{-1}{x+2} + \frac{3}{x+3}$

$= \frac{-1}{2(1-(-\frac{x}{2}))} + \frac{3}{3(1-(-\frac{x}{3}))} = \frac{-1}{2}\frac{1}{1-(-\frac{x}{2})} + \frac{1}{1-(-\frac{x}{3})}$

$= -\frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n + \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{3}\right)^n$

$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2^{n+1}} x^n + \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n} x^n$

$= \sum_{n=0}^{\infty} \left[\left(\frac{-1}{2}\right)^{n+1} + \left(\frac{-1}{3}\right)^n\right] x^n$

Ex 6

$\int_0^{1/2} \arctan(x/2)\,dx = \int_0^{1/2} \left(\int \frac{d}{dx}\arctan\left(\frac{x}{2}\right)dx\right)dx$

$\cancel{\frac{d}{dx}\tan^{-1}\left(\frac{x}{2}\right) = \frac{1}{2}\frac{1}{1+x^2} = \frac{1}{2}\frac{1}{1-(-x^2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n}}$

$\cancel{\tan^{-1}\left(\frac{x}{2}\right) = \int \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}}$

Ex 6 Approximate $\int_0^{1/2} \tan^{-1}\left(\frac{x}{2}\right) dx$ using power series.

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{2}\right) = \frac{1}{2}\frac{1}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{2n}x^{2n} = \sum(-1)^n\left(\frac{1}{2}\right)^{2n+1}x^{2n}$$

$$\Rightarrow \tan^{-1}\left(\frac{x}{2}\right) = \int \sum(-1)^n\left(\frac{1}{2}\right)^{2n+1}x^{2n}\,dx$$

$$= \sum(-1)^n\left(\frac{1}{2}\right)^{2n+1}\frac{1}{2n+1}x^{2n+1}$$

So we want $\int_0^{1/2} \sum(-1)^n\left(\frac{1}{2}\right)^{2n+1}\frac{1}{2n+1}x^{2n+1}\,dx$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{2n+1}\frac{1}{2n+1}\frac{1}{2n+2}x^{2n+2}\Big|_0^{1/2}$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{4n+3}\frac{1}{2n+1}\frac{1}{2n+2}$$

Using error for Alternating series $n=2$

$$\text{error} < \left(\frac{1}{2}\right)^{4\cdot 3+3}\frac{1}{2\cdot 3+1}\frac{1}{2\cdot 3+2} \sim .0000005$$

$$\Rightarrow \int_0^{1/2}\tan^{-1}\left(\frac{x}{2}\right)dx \approx \left(\frac{1}{2}\right)^3\cdot\frac{1}{2} - \left(\frac{1}{2}\right)^7\cdot\frac{1}{3}\cdot\frac{1}{4} \approx .061848$$