Table from 11.10

$$\frac{1}{1-x} = \sum x^n \qquad e^x = \sum \frac{x^n}{n!}$$

$$\sin(x) = \sum (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos(x) = \sum (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan^{-1} x = \sum (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$(1+x)^k = \sum \binom{k}{n} x^n$$

Binomial series

Binomial coefficient: $\binom{k}{n} = \frac{k!}{(k-n)!\,n!} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$

Binomial Theorem: $(x+y)^k = \sum_{n=0}^{k} \binom{k}{n} x^n y^{k-n}$

Ex1 $(x+y)^5 = \binom{5}{0} x^5 y^0 + \binom{5}{1} x^4 y + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} x y^4 + \binom{5}{5} x^0 y^5$

$= x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$

Pascal's Triangle

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

Binomial Series: If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

To see this, $f(x) = (1+x)^k$

$$f'(x) = k(1+x)^{k-1}$$
$$f''(x) = k(k-1)(1+x)^{k-2}$$
$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$$
$$\vdots$$
$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$$

$$f(0) = 1$$
$$f'(0) = k$$
$$f''(0) = k(k-1)$$
$$f'''(0) = k(k-1)(k-2)$$
$$\vdots$$
$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$$

$\Rightarrow$ Maclaurin series given by $\sum \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$

Ex 2 $\sqrt[4]{1-x}$

$$= (1-x)^{1/4} = \sum_{n=0}^{\infty} \binom{1/4}{n} (-1)^n x^n$$
$$= 1 - \frac{1}{4}x + \frac{(-1)^2 \, 1/4 \, (-3/4)}{2!} x^2 + \frac{(-1)^3(\frac{1}{4})(-\frac{3}{4})(-\frac{7}{4})}{3!} x^3 + \cdots + \frac{(-1)^n(\frac{1}{4})(-\frac{3}{4})\cdots(\frac{1}{4}-n+1)}{n!} x^n + \cdots$$
$$= 1 - \frac{1}{4}x - \frac{\cancel{(-1)}\cdot 1\cdot 3}{2!\,4^2} x^2 - \frac{1\cdot 3\cdot 7}{3!\,4^3} x^3 + \cdots - \frac{\cancel{(-1)^{n-1}}\cdot 1\cdot 3\cdot 7\cdots \overset{(4n-5)}{\cancel{(n+4)}}}{n!\;4^n} x^n + \cdots$$
$$= 1 - \frac{1}{4}x - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot \cdots \cdot (4n-5)}{n!\;4^n} x^n$$

Ex 3 $\frac{1}{(2+x)^3}$

$$= \frac{1}{[2(1+\frac{x}{2})]^3} = \frac{1}{8} \frac{1}{(1+\frac{x}{2})^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$$

$$= \frac{1}{8} \sum_{n=0}^{\infty} \binom{-3}{n} \left(\frac{x}{2}\right)^n = \frac{1}{8}\Big[1 + \left(\frac{-3}{1}\right)\frac{x}{2} + \frac{(-3)(-4)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(\frac{x}{2}\right)^3$$
$$+ \cdots + \frac{(-3)(-4)(-5)\cdots (-1)^n \cancel{(3)(4)}\cdot(n+2)}{n!} \left(\frac{x}{2}\right)^n \Big]$$

$$= \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n\, 3\cdot 4\cdot 5\cdots (n+2)}{n!\;2^n} x^n$$

$$= \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdots (n)(n+1)(n+2)}{n!\cdot 2\cdot 2^n} x^n$$

$$= \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2^{n+1}} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2^{n+4}} x^n$$

Ex 4 $x\cos(\frac{1}{2}x^2)$. Using, $\cos x = \sum (-1)^n \frac{x^{2n}}{(2n)!}$

$$x\cos(\tfrac{1}{2}x^2) = x\sum(-1)^n \frac{[\frac{1}{2}x^2]^{2n}}{(2n)!}$$

$$= x\sum(-1)^n \frac{(\frac{1}{2})^{2n}x^{4n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n)!\,2^{2n}}$$

Ex 5 $e^{3x} - e^{2x}$

$$= \sum_{n=0}^{\infty}\frac{(3x)^n}{n!} - \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{3^n - 2^n}{n!}x^n$$

Ex 6 $\sin^2 x$

$$= \tfrac{1}{2}(1-\cos 2x) = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x = \tfrac{1}{2} - \tfrac{1}{2}\sum(-1)^n\frac{(2x)^{2n}}{(2n)!}$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\sum(-1)^n\frac{2^{2n}x^{2n}}{(2n)!} = \tfrac{1}{2} - \sum_{n=0}^{\infty}\frac{2^{2n-1}x^{2n}}{(2n)!}$$

Warning Writing $\sin^2 x$ as $\left[\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right]^2$ is n't helpful because we ~~have~~ dont know how to square a series.

Ex 7 $\int x^2 \sin x^2\, dx = \int x^2 \sum(-1)^n \frac{x^{2n+1}}{(2n+1)!}\, dx$

$$\sum\int(-1)^n\frac{x^{2n+3}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{(2n+4)(2n+1)!}$$

Using series to evaluate limits

Ex 8 $\lim_{x\to 0} \frac{x-\ln(1+x)}{x^2} = \lim_{x\to 0} \frac{x - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}{x^2}$

$= \lim_{x\to 0} \frac{x-(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots)}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\cdots}{x^2}$

$= \lim_{x\to 0} \frac{x^2(\frac{1}{2}-\frac{x}{3}+\frac{x^2}{4}-\cdots)}{x^2} = \frac{1}{2}$

Ex 9 $\lim_{x\to 0} \frac{x^3-3x+3\tan^{-1}x}{x^5} = \lim_{x\to 0} \frac{x^3-3x+3(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots)}{x^5}$

$= \lim_{x\to 0} \frac{\frac{3}{5}x^5-\frac{3}{7}x^7+\frac{3}{9}x^9-\cdots}{x^5} = \frac{3}{5}$