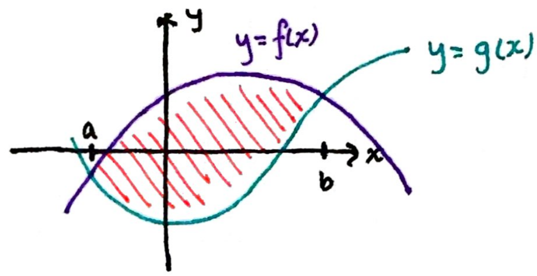
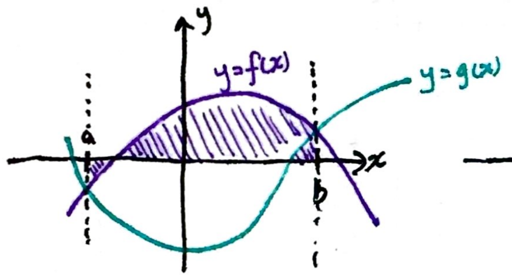
6.1 Area between curves

Recall: $\int_a^b f(x)\,dx$ represents the signed area between $y=f(x)$ and the x-axis.

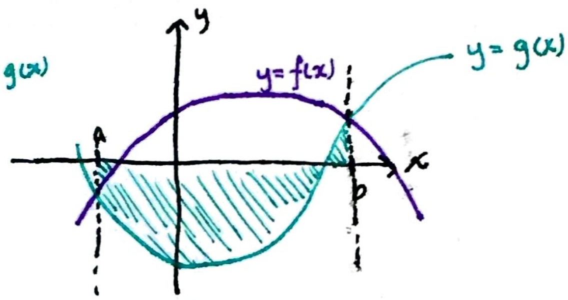
Goal: To calculate area between two curves



Break this up into what we know how to do



$\int_a^b f(x)\,dx$



$\int_a^b g(x)\,dx$

Notice that if we subtract these two areas, we get exactly the area we want.

Thus, the area A of the region bounded by the curves $y=f(x)$ and $y=g(x)$, and the vertical lines $x=a$ and $x=b$, where f and g are continuous with $f(x) \geq g(x)$ on $[a,b]$ is

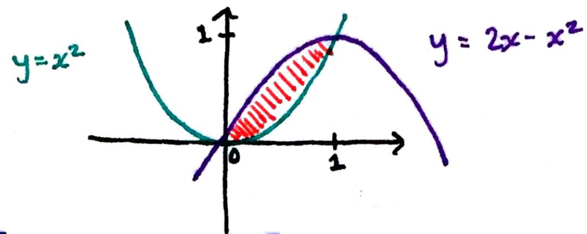
$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx} \qquad (1)$$

Remark We can lift the restriction $f(x) \geq g(x)$ on $[a,b]$ with a slight modification to (1). We can do this by taking absolute values, so we're effectively always taking the larger function minus the smaller one.

$$\boxed{A = \int_a^b |f(x) - g(x)|\,dx} \qquad (2)$$

Ex 1 Find the area of the region bounded by $y = x^2$ and $y = 2x - x^2$.

Note Always draw a picture!



We want $\int [\text{Top} - \text{Bottom}]$

$$\int_0^1 [(2x - x^2) - x^2]\, dx = \int_0^1 [2x - 2x^2]\, dx$$

pts of intersection

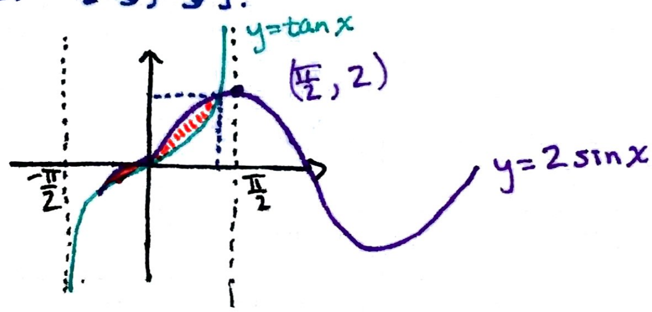
$$= x^2 - \tfrac{2}{3}x^3 \Big|_0^1$$

$$= 1 - \tfrac{2}{3}$$

$$= \tfrac{1}{3}$$

Ex 2 Find the area between $y = \tan x$ and $y = 2\sin x$ on the interval $[\frac{-\pi}{3}, \frac{\pi}{3}]$.

This should look symmetric about the y-axis.



The two curves intersect at $(0,0)$. We need to figure out the other point of intersection. We do this by setting the two eqns equal to one another.

$$\tan x = 2\sin x$$

$$\frac{\cancel{\sin x}}{\cos x} = 2\cancel{\sin x} \qquad \text{We've already considered } \sin x = 0.$$

$$\tfrac{1}{2} = \cos x$$

$$\boxed{x = \tfrac{\pi}{3}}$$

So the upper bound for the integral is $\frac{\pi}{3}$, and the area of the enclosed region is given by

$$A = \int_0^{\frac{\pi}{3}} [2\sin x - \tan x]\, dx$$

$$= -2\cos x\Big|_0^{\pi/3} + \int_1^{\frac{1}{2}} \frac{du}{u}$$

$u = \cos x$
$du = -\sin x\, dx$

$$= -2\cos x\Big|_0^{\pi/3} + \ln u\Big|_1^{1/2}$$

$$= [-2(\tfrac{1}{2}) + 2(1)] + [\ln \tfrac{1}{2} - \ln 1]$$

$$= 1 + \ln \tfrac{1}{2}$$

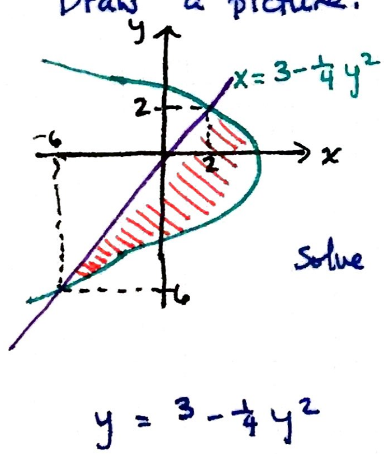
$$= 1 - \ln 2$$

But this is only half the area. We could repeat this process for the area between the curves on $[-\frac{\pi}{3}, 0]$, or use symmetry to conclude

$$\boxed{A = 2 - 2\ln 2}.$$

Question Find the area bounded by the curves $4x + y^2 = 12$ and $x = y$.

Draw a picture!



Here, x is a function of y:

$$4x + y^2 = 12 \iff x = 3 - \tfrac{1}{4}y^2$$

To find the pts of intersection, we solve the system

$$\begin{cases} x = y \\ x = 3 - \frac{1}{4}y^2 \end{cases}$$

$$y = 3 - \tfrac{1}{4}y^2$$

$$4y = 3 - y^2$$

$$y^2 + 4y - 12 = 0$$

$$(y+6)(y-2) \implies \boxed{y = -6, 2}$$

Plugging this in to $y=x$, we get $(-6,-6)$ and $(2,2)$ as the pts of intersection.

Here we can't apply $\int[\text{Top} - \text{Bottom}]$. But recall from (1) that this just means $\int[\text{larger} - \text{smaller}]$. So with x as a function of y, this gives us

$$\boxed{A = \int [\text{Right} - \text{Left}]}$$

So the area for this problem is given by

$$A = \int_{-6}^{2} \left[\left(3 - \tfrac{1}{4}y^2\right) - y\right] dy$$

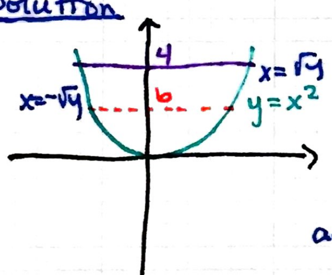
$$= 3y - \tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 \Big|_{-6}^{2}$$

$$= \left[3(2) - \tfrac{1}{12}(2)^3 - \tfrac{1}{2}(2)^2\right] - \left[3(-6) - \tfrac{1}{12}(-6)^3 - \tfrac{1}{2}(-6)^2\right]$$

$$= \boxed{\tfrac{64}{3}}.$$

<u>Problem</u> Find the number b such that the line $y=b$ divides the region bounded by the curves $y=x^2$ and $y=4$ into two regions with equal area.

<u>Solution</u>



By symmetry, we can restrict ourselves to the first quadrant. Since we want to find b such that the area is split in half, we want the two parts to have equal area. This problem becomes pretty easy if we think "dy". We want

$$\int_0^b \sqrt{y}\, dy = \int_b^4 \sqrt{y}\, dy$$

$$\int_0^b y^{1/2}\, dy = \int_b^4 y^{1/2}\, dy$$

$$\tfrac{2}{3} y^{3/2} \Big|_0^b = \tfrac{2}{3} y^{3/2} \Big|_b^4$$

$$\tfrac{2}{3} b^{3/2} = \tfrac{2}{3}\left(4^{3/2} - b^{3/2}\right)$$

$$\tfrac{4}{3} b^{3/2} = \tfrac{2}{3} \cdot 4^{3/2}$$

$$2 b^{3/2} = 8$$

$$b^{3/2} = 4$$

$$\boxed{b = 4^{2/3}}$$

Remark It may be necessary to break an integral into smaller parts that we know how to do.

Summary To find area between two curves:

1) Determine whether there is a Top & Bottom function or Left and Right function. If not, break down the region to achieve this.
2) Top / Bottom: $\int_a^b [Top - Bottom]\, dx$ gives area
   Left / Right: $\int_a^b [Right - Left]\, dy$ gives area.
3) To find points of intersection, set equations equal to one another