10.2 Parametric Curves.

f, g differentiable, want to find tangent line at a point on curve $x=f(t)$, $y=g(t)$, y also differentiable function of x. Then by the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, we can solve to get $\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$.

From this equation, it follows that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

Warning! $\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$.

Ex1 $x = t\sin t$, $y = t^2+t$

Then $\frac{dy}{dt} = 2t+1$, $\frac{dx}{dt} = t\cos t + \sin t$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2t+1}{t\cos t + \sin t}}$$

Ex2 Find equation of tangent line at $t=0$ for $x = e^t \sin\pi t$, $y = e^{2t}$

$\frac{dy}{dt} = 2e^{2t}$

$\frac{dx}{dt} = \pi e^t \cos\pi t + e^t \sin\pi t$

$\Rightarrow \frac{dy}{dx} = \frac{2e^{2t}}{\pi e^t\cos\pi t + e^t\sin\pi t}$

at $t=0$, $\frac{dy}{dx} = \frac{2}{\pi}$

When $t=0$, $x = e^0 \sin 0 = 0$, $y = e^{2\cdot 0} = 1$

So tangent line is given by $y = \frac{2}{\pi}(x-0)+1$
$= \frac{2}{\pi} + 1$

Ex3 Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.. For what t is curve concave up?

$x = t^2+1$, $y = e^t - 1$

$\frac{dy}{dt} = e^t$, $\frac{dx}{dt} = 2t \rightarrow \frac{dy}{dx} = \frac{e^t}{2t}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{e^t}{2t}\right)}{2t} = \frac{-2t^{-2}e^t + 2e^t t^{-1}}{2t}$

Setting $\frac{d^2y}{dx^2} = 0$, $-\cancel{4}t^{-3}\cdot e^t + \cancel{2}e^t \cdot t^{-2} = 0$

$\cancel{-2e^t + 2te^t = 0}$

$\cancel{2e^t(-1+t) = 0}$

$\cancel{\boxed{t=1}}$

$t^{-3}e^t(-1+t) = 0 \Rightarrow t=1$

Plugging in $t=2$, get $-2^{-3}e^2 + e^2 \cdot 2^{-2} = 2^{-3}e^2(-1+2) > 0$

$\Rightarrow$ concave up on $(+1, \infty)$

Arc length

From chapter 8, we know that if a curve C is given by $y = F(x)$ and $F'(x)$ is continuous, $a \le x \le b$, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If we can parametrize C by $x = f(t)$, $y = g(t)$ for $\alpha \le t \le \beta$ with $\frac{dx}{dt} > 0$, then since $f(\alpha) = a$, $f(\beta) = b$, we have

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$\boxed{L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Ex 4 Find the exact length

$x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.

$\left(\frac{dx}{dt}\right)^2 = (6t)^2 = 36t^2$, $(dy/dt)^2 = (6t^2)^2 = 36t^4$

$$L = \int_0^1 \sqrt{36t^2 + 36t^4}\, dt = 6\int_0^1 t\sqrt{1 + t^2}\, dt$$

$u = 1 + t^2$
$du = 2t\,dt$

$$= 3\int_1^2 u^{1/2}\, du = 3\left(\frac{2}{3}u^{3/2}\right)\Big|_1^2$$

$$= 2 \cdot \left(2^{3/2} - 1\right)$$

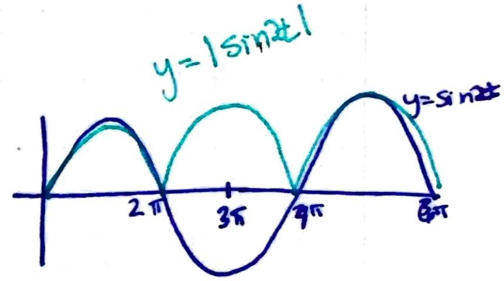
Ex 5 Find the distance traveled by particle with position (x,y) as t varies. Compare with length of curve.

$$x = \sin^2 t, \quad y = \cos^2 t, \qquad 0 \le t \le 3\pi.$$

$$\frac{dx}{dt} = 2\sin t\cos t, \qquad \frac{dy}{dt} = -2\cos t\sin t$$

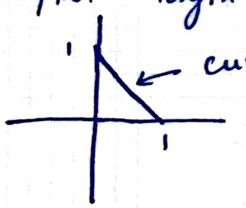
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\sin^2 t\cos^2 t + 4\sin^2 t\cos^2 t$$
$$= 8\sin^2 t\cos^2 t$$
$$= 2\sin^2(2t)$$

So the distance traveled is given by



$$D = \int_0^{3\pi} \sqrt{2}\,|\sin(2t)|\,dt$$
$$= 6\sqrt{2}\int_0^{\pi/2} \sin(2t)\,dt$$
$$= 6\sqrt{2}$$

To find length of curve, notice that $x, y \ge 0$ and $x+y=1$.



curve given

Whole curve is traversed as t goes from 0 to $\frac{\pi}{2}$,

So $L = \int_0^{\pi/2} \sqrt{2}\,\sin(2t)\,dt = \sqrt{2}$.