App H Complex Numbers

A complex number is of the form $a+bi$, where $a,b \in \mathbb{R}$, where i has the property that $i^2=-1$. We can think of complex numbers as ordered pairs in the plane (a,b) and $i=(0,1)$.

a is called the real part of $a+bi$,
b is called the imaginary part of $a+bi$.

$a+bi$ and $c+di$ are equal if $a=b$ and $c=d$.
We define multiplication and addition as follows:

$$(a+bi)(c+di) = ac + adi + bci + bdi^2$$
$$= (ac-bd) + (ad+bc)i$$

$$(a+bi) + (c+di) = (a+b) + (c+d)i$$

The complex conjugate of $a+bi$, $\overline{a+bi} = a-bi$.

Properties

$$\overline{z+w} = \bar{z}+\bar{w}, \quad \overline{zw} = \bar{z}\,\bar{w}, \quad \overline{z^n} = (\bar{z})^n$$

The modulus of $z=a+bi$ is $|z| = \sqrt{a^2+b^2}$

Notice $z\bar{z} = (a+bi)(a-bi) = a^2+b^2 = |z|^2$

Ex 1 Evaluate $(2+3i)(1-4i)$

$$(2+3i)(1-4i) = 2-8i+3i-12i^2 = 14-5i$$

Ex2/ Evaluate $\frac{4+8i}{9+4i}$

Notice that $\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z\bar{w}}{|w|^2}$

So if we multiply the top and bottom by the complex conjugate of the bottom we will have something of the form $a+bi$.

$$\frac{4+8i}{9+4i} \cdot \frac{9-4i}{9-4i} = \frac{36-16i+72i-32i^2}{81+16} = \frac{68+56i}{97}$$

$$= \frac{68}{97} + \frac{56}{97}i$$

Polar Form

$z = r\cos\theta + i r\sin\theta$ (same as polar coords)

θ is called the argument of z, written $\arg z$.

Notice that $r = |z|$.

Multiplying numbers in Polar form

$z_1 = r_1\cos\theta_1 + i r_1 \sin\theta_1$, $z_2 = r_2\cos\theta_2 + i r_2\sin\theta_2$

then $z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

Euler's Formula $re^{i\theta} = r\cos\theta + i r\sin\theta$

So we could write $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, n a positive integer,

then $z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = r^n e^{in\theta}$

Ex 3 $\sqrt{-2}\,\sqrt{-2} = i\sqrt{2}\cdot i\sqrt{2} = i^2\cdot 2 = -2$

Ex 4 $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}$

Use polar form. $r = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$, $\tan\theta = \frac{1/2}{1/2} = 1 \Rightarrow \theta = \pi/4$

So $\left(\frac{1}{2}+\frac{1}{2}i\right)^{10} = \left[\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{10}$

$= \left(\frac{1}{\sqrt{2}}\right)^{10}\left(\cos\left(\frac{10\pi}{4}\right)+i\sin\left(\frac{10\pi}{4}\right)\right)$

$= \frac{1}{32}\left(\cos\left(\frac{5\pi}{2}\right)+i\sin\left(\frac{5\pi}{2}\right)\right)$

$= \frac{i}{32}$

Ex 5 Evaluate i^{692}

Note that $i = i$, $i^2=-1$, $i^3 = i\,i^2 = -i$, $i^4 = 1$, $i^5 = i$.
So i^n is periodic with period 4.
Since 692 is divisible by 4, $i^{692} = (i^4)^{173} = (1)^{173} = 1$.

Fundemental Theorem of Algebra

A polynomial of degree n with complex coefficents has n roots. This means that every polynomial can be factored as a product of linear factors.

Ex 6 Find all solutions to

$x^2+x+1=0.$

Quadratic formula! ~~or~~

$$x = \frac{-1\pm\sqrt{1^2-4(1)(1)}}{2} = \frac{-1\pm i\sqrt{3}}{2} = \frac{-1+i\sqrt{3}}{2},\ \frac{-1-i\sqrt{3}}{2}$$

Ex 7 Find all sixth roots of $z = -8$

We want to find all w such that $w^6 = -8$

If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, then z has n distinct nth roots.

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] = r^{1/n} e^{i\left(\frac{\theta+2\pi k}{n}\right)}$$

for $k = 0, 1, \ldots, n-1$.

Using this fact, $-8 = 8e^{i\pi}$

$$w_k = 8^{1/6} e^{i\left(\frac{\pi + 2\pi k}{6}\right)}, \quad 0 \le k \le 5$$

That is,

$$8^{1/6} e^{i\pi/6} = \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$8^{1/6} e^{i(\pi/6 + \frac{2\pi}{6})} = 8^{1/6} e^{i\frac{\pi}{2}} = i\sqrt{2}$$
$$8^{1/6}\left(e^{i\left(\frac{\pi}{6} + \frac{4\pi}{6}\right)}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$
$$8^{1/6}\left(e^{i\left(\frac{\pi}{6} + \frac{6\pi}{6}\right)}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}\right)$$
$$8^{1/6}\left(e^{i(\pi/6 + \frac{8\pi}{6})}\right) = \sqrt{2}(-i) = -i\sqrt{2}$$
$$8^{1/6}\left(e^{i(\frac{\pi}{6} + \frac{10\pi}{6})}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}\right)$$