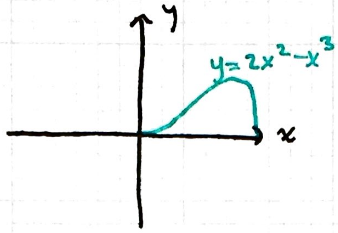
6.3 Volumes by cylindrical shells.

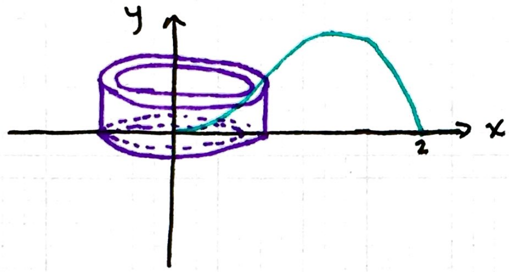
Recall From 6.2 you should know how to find volumes using the disk/washer methods and by using cross sections.

Goal Find volume of solid obtained by rotating about the y-axis the region bounded by $y=2x^2-x^3$ and $y=0$.

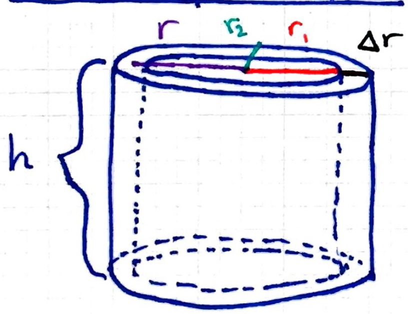


Washer method isn't good because it would force us to solve $2x^2-x^3=y$ for x.

Idea: Cylindrical shells



Volume of a shell



Let V_2 be the volume of outer cylinder, V_1 the inner cylinder's volume.

Then the volume of the shell is

$$V = V_2 - V_1.$$

$$= \pi r_2^2 h - \pi r_1^2 h$$

$$= \pi(r_2^2 - r_1^2)h$$

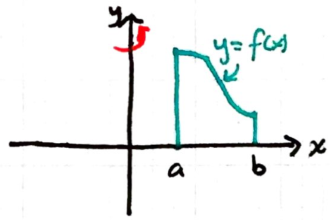
$$= \pi(r_2+r_1)(r_2-r_1)h$$

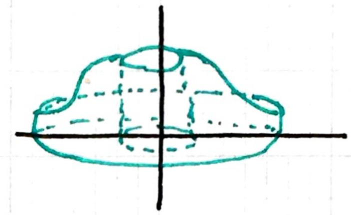
$$= 2\pi \underbrace{\frac{r_2+r_1}{2}}_{r} \underbrace{(r_2-r_1)}_{\Delta r} h$$

$$\boxed{V = 2\pi r h \Delta r} \quad (1)$$

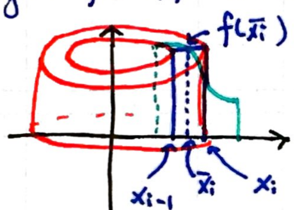
V = [circumference][height][thickness]

Like when we did Riemann sums. Consider the solid S obtained by rotating about the y-axis the region bounded by $y=f(x)$, $y=0$, $x=a$ and $x=b$, where $f(x)\geq 0$, $b>a\geq 0$.





1) Divide $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ of width Δx, and let $\bar{x}_i$ be the midpt of the ith subinterval
2) Rotating the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ about the y-axis gives a shell with avg radius $\bar{x}_i$, height $f(\bar{x}_i)$, thickness Δx.



Using (1), this shell has volume

$$V_i = (2\pi \bar{x}_i)[f(\bar{x}_i)]\,\Delta x$$

So $V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\,\Delta x$

Taking limit as $n\to\infty$ gives us precisely $\int_a^b 2\pi x f(x)\,dx$

To summarize, using the shell method: the volume of a solid obtained by rotating about the y-axis the region under a non-negative curve $y=f(x)$ from a to b is

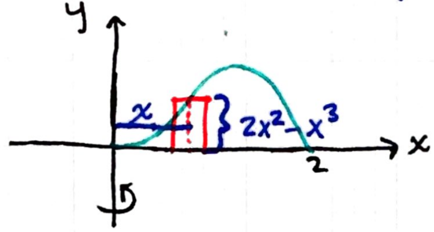
$$\boxed{V = \int_a^b 2\pi x\, f(x)\,dx} \qquad (2)$$

The formula (2) should be remembered as

$$\int \underbrace{[\text{circumference}]}_{2\pi x}\underbrace{[\text{height}]}_{f(x)}\underbrace{[\text{thickness}]}_{dx}$$

Back to our goal...

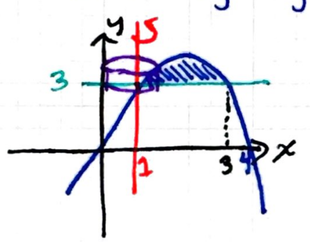
Ex1 Volume of solid obtained by rotating about y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



$$V = \int_0^2 2\pi x\,(2x^2 - x^3)\,dx$$
$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$$
$$= \boxed{\frac{16}{5}\pi}$$

Can do this with other axes of rotation as well.

Ex2 Volume of region obtained by rotating about the line $x=1$ region bounded by $y = 4x - x^2$ and $y = 3$.



In this case, instead of having a radius of x, we have a radius of $x-1$ since we're starting at $x=1$.

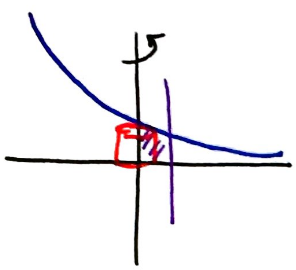
Moreover, our shell starts at $y=3$ instead of $y=0$, giving a height of $f(x) - 3 = (4x - x^2) - 3$.

So the volume is given by

$$\int_1^3 \underbrace{2\pi(x-1)}_{\text{circum.}}\underbrace{(-x^2+4x-3)}_{\text{height}}\underbrace{dx}_{\text{thickness}}$$

Problem 1. Use the method of cylindrical shells to find the volume V generated by rotating the region bounded by the given curves about the y-axis.

$$y = 7e^{-x^2}, \quad y = 0, \quad x = 0, \quad x = 1$$



radius : x

height: $f(x) = 7e^{-x^2}$

$$\int_0^1 2\pi x \cdot 7e^{-x^2}\, dx$$

$$= 7\cancel{4}\pi \int_0^1 2x\, e^{-x^2}\, dx$$

$u = -x^2$

$du = -2x\,dx$

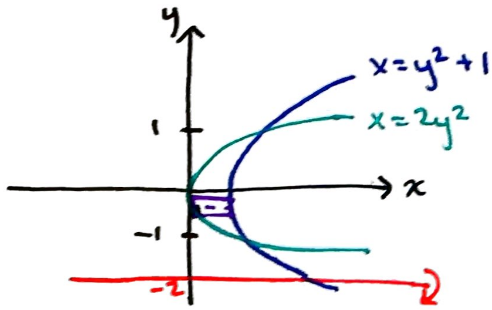
$$= -7\pi \int_0^{-1} e^u\, du$$

$$= 7\pi\, e^u \Big|_{-1}^{0}$$

$$\boxed{= 7\pi\left(1 - \frac{1}{e}\right)}$$

Problem 2. Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curve about the specified axis.

(a) $x = 2y^2$, $x = y^2 + 1$; about $y = -2$

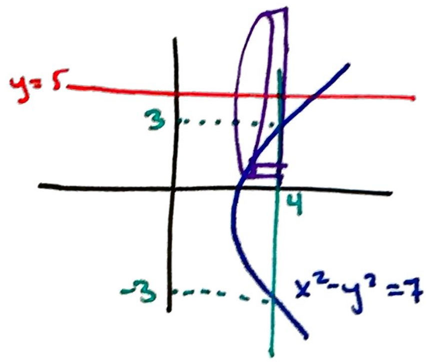


radius $y - (-2) = y+2$

height $y^2 + 1 - 2y^2 = 1 - y^2$

$$\boxed{V = \int_{-1}^{1} 2\pi (y+2)(1-y^2)\, dy}$$

(b) $x^2 - y^2 = 7$, $x = 4$; about $y = 5$



radius $5-y$

height $4 - x = 4 - \sqrt{7+y^2}$

$$\boxed{V = \int_{-3}^{3} 2\pi (5-y)\left(4 - \sqrt{7+y^2}\right)\, dy}$$