Physics things

Newton's Second Law of Motion: $F = ma$, where m is mass of object, a is acceleration.

Metric Force measured in Newtons ($N = kg \cdot m/s^2$)
US Force measured in lbs.

Work

$W = Fd$: Work = Force · distance

Metric Work measured in N-m (Joules)
US Work measured in ft-lbs. (1 ft-lb ~ 1.36 J)

Ex 1 How much work is done when a hoist lifts a 200-kg rock to a height of 3m?

Solution Force on rock is equal and opposite to force exerted by gravity. Newton ⟹

$$F = (200)(9.8)$$
$$= 1960 \ N$$

[acceleration due to gravity ~ 9.8 m/s²]

So $W = (1960 N)(3m) = \boxed{5880 \ J.}$

Remark Given a weight in pounds, we don't have to account for acceleration due to gravity because the weight of an object IS the force acting on the object.

What if F isn't constant?

Suppose an object moves along x-axis from a to b with a variable force $f(x)$, which is a continuous function of x on $[a, b]$.

Idea break up $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ so that each subinterval's force can be approximated by a chosen sample x_i^*.

If the width of each $[x_{i-1}, x_i]$ is Δx, then the work done moving the object from x_{i-1} to x_i is approximated by

$$W_i \approx f(x_i^*)\,\Delta x \qquad (\text{force} \times \text{distance})$$

Adding up all the subintervals, the total work

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} f(x_i^*)\,\Delta x .$$

This is precisely a Riemann sum. Letting $n \to \infty$, we get

$$\boxed{W = \int_a^b f(x)\,dx} \qquad (1)$$

Ex 2 A variable force of $5x^{-2}$ pounds moves an object along a straight path when it is x feet from the origin. Calculate the work done in moving the object from $x=1$ ft to $x=10$ft.

Solution Using (1),
$$\begin{aligned} W &= \int_1^{10} 5x^{-2}\,dx \\ &= -5x^{-1}\Big|_1^{10} \\ &= \frac{-5}{10} + 5 \\ &\boxed{= 4.5 \text{ ft-lbs.}} \end{aligned}$$

Hooke's Law

The force required to maintain a spring stretched x units beyond its natural length is proportional to x; ie,

$$f(x) = kx,$$

where k is a positive constant (called spring constant).

Ex3 A spring has a natural length of 40 cm. If a 60-N force is required to keep the spring compressed 10 cm, how much work is done during this compression? How much work is required to compress the spring to a length of 25cm?

Solution Hooke's Law also applies to compression. So to compress by .1 m (10 cm) requires 60-N force $\Rightarrow$

$$f(.1) = 60$$
$$.1k = 60$$
$$\boxed{k = 600}$$

So the work done to compress the spring is

$$W = \int_0^{.1} 600x\,dx$$
$$= 300x^2 \Big|_0^{.1}$$
$$= \boxed{3\text{ J}}$$

Finally, to compress to length of 25 cm is to compress by .15 m, and work done is

$$W = \int_0^{.15} 600x\,dx$$
$$= 300x^2 \Big|_0^{.15}$$
$$= \boxed{6.75\text{ J.}}$$

Ex4 A thick cable 60 ft long, weighing 180 lbs hangs from a winch on a crane. Compute the work done if the winch windus up 25 ft of the caple.

Solution Winding up x ft of cable $\Rightarrow$ 60-x ft of cable still hanging. The weight of what's hanging is

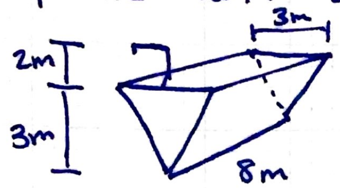
$$\frac{180\text{ lbs}}{60\text{ ft}} \cdot ((60-x)\text{ ft} = 3(60-x)\text{ lbs.}$$

Lifting by Δx ft $\Rightarrow$ $3(60-x)\Delta x$ ft-lb of work.

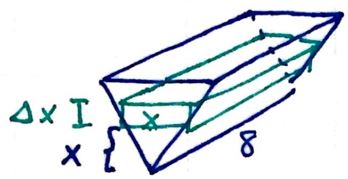
Total work

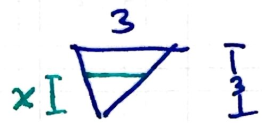
$$\int_0^{25} 3(60-x)\,dx = \boxed{3562.5\text{ ft-lb.}}$$

Ex5 A tank is full of water. Find the work required to pump the water out of the spout.



Consider a slice.





Sim triangles ⇒ smaller △ has base of x.

So volume of a slice: $\boxed{8x\,\Delta x}$

Mass = (density)(vol) $= \boxed{8 \cdot 10^3\, x\,\Delta x}$

Must overcome force of gravity:

$W = Fd = mgd = 9.8 \cdot 10^3 \cdot 8x\,\Delta x\,\underbrace{(5-x)}_{\text{distance to be raised}}$

Now total work done

$$\int_0^3 (9.8)10^3\, 8x(5-x)\,dx$$

$$\boxed{\sim 1.06 \cdot 10^6 \text{ J.}}$$

6.5 Avg value

Idea Discrete case: Given n numbers $x_1, \dots, x_n$, the average is given by

$$\frac{1}{n}\sum_{i=1}^{n} x_i .$$

The continuous case is defined similarly. Given f continuous on $[a,b]$, the average value of f on $[a,b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Ex 6 Find the average value of $f(x) = 1+x^2$ on $[-1,2]$

$$f_{ave} = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx$$

$$= \boxed{2}$$

Mean Value Thm for integrals

If f is continuous on $[a,b]$ then there exists $c \in [a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx .$$

Proof Follows from MVT for derivatives and FTC.
Let $F(x) = \int_a^x f(t)\,dt$. Then MVT $\Rightarrow$ there exists c s.t.

$$F'(c) = \frac{F(b)-F(a)}{b-a}.$$

But $F'(c) = f(c)$, $F(b) = \underbrace{\int_a^b f(t)\,dt}_{\text{same as } \int_a^b f(x)\,dx}$, $F(a) = 0$

Conclude: $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$ $\boxtimes$