Basic trig integrals

Ex1

$$\int \sin^5 x \, dx$$

Solution $u = \sin x$ not helpful. We make use of fact that

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

$$\int \sin^5 x \, dx = \int \sin x \sin^4 x \, dx$$

$$= \int \sin x (\sin^2 x)^2 \, dx$$

$$= \int \sin x (1-\cos^2 x)^2 \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \int \sin x (1 - 2\cos^2 x + \cos^4 x) \, dx$$

$$= \int \sin x \, dx - 2\int \sin x \cos^2 x \, dx + \int \sin x \cos^4 x \, dx$$

$$= \int \sin x \, dx + 2\int u^2 \, du - \int u^4 \, du$$

$$\boxed{= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x \, dx + C}$$

Ex2

$$\int \sin^5 x \cos^4 x \, dx = \int (\sin^2 x)^2 \cos^4 x \sin x \, dx$$

$$= \int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx$$

$u = \cos x \, dx$

$du = -\sin x \, dx$

$$= -\int (1-u^2)^2 u^4 \, du$$

$$= -\int (1 - 2u^2 + u^4) u^4 \, du$$

$$= -\int (u^4 - 2u^6 + u^8) \, du$$

$$= -\left(\tfrac{1}{5}u^5 - \tfrac{2}{7}u^7 + \tfrac{1}{9}u^9\right) + C$$

$$\boxed{= -\left(\tfrac{1}{5}\cos^5 x - \tfrac{2}{7}\cos^7 x + \tfrac{1}{9}\cos^9 x\right) + C}$$

Sometimes need half-angle formulas

$$\boxed{\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)}$$

Ex 3

$$\int \cos^4 x\, dx = \int (\cos^2 x)^2\, dx$$

$$= \int \left[\tfrac{1}{2}(1+\cos 2x)\right]^2 dx$$

$$= \tfrac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$$

$$= \tfrac{1}{4}\int \left(1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)\right) dx$$

$$= \tfrac{1}{4}\left(x + \sin 2x + \tfrac{1}{2}x + \tfrac{1}{8}\sin 4x\right) + C$$

$$\boxed{= \tfrac{1}{4}\left(\tfrac{3}{2}x + \sin 2x + \tfrac{1}{8}\sin 4x\right) + C}$$

Practice

1) $\int \sec^3 x\, dx$ (May use $\int \sec x\, dx = \ln|\sec x + \tan x| + C$)

2) $\int \tan^2 x \sec^4 x\, dx$

1) $\int \sec^3 x\, dx = \int \sec^2 x \sec x\, dx$

$u = \sec x \qquad dv = \sec^2 x\, dx$

$du = \sec x \tan x \qquad v = \tan x$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \ln|\sec x + \tan x| + C$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x\, dx = \boxed{\tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C}$$

2) $\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x \, dx$

$$= \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx \qquad u = \tan x, \quad du = \sec^2 x \, dx$$

$$= \int u^2 (1 + u^2) \, du$$

$$= \int (u^2 + u^4) \, du$$

$$= \frac{1}{3} u^3 + \frac{1}{5} u^5 + C$$

$$\boxed{= \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C}$$

7.3 Trig Substitutions

Ex4 $\int \frac{dx}{x^2 \sqrt{4 - x^2}}$ $x = 2 \sin\theta$, $dx = 2\cos\theta \, d\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

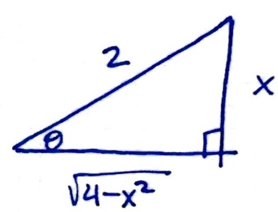
$$= \int \frac{2\cos\theta \, d\theta}{4 \sin^2\theta \sqrt{4(1 - \sin^2\theta)}}$$

$$= \frac{\cancel{2}}{8} \int \frac{\cancel{2}\cos\theta}{\sin^2\theta \cos\theta} \, d\theta$$

$$= \frac{1}{4} \int \csc^2\theta \, d\theta$$

$$= -\frac{1}{4} \cot\theta + C$$

$$= \boxed{-\frac{1}{4} \cdot \frac{\sqrt{4 - x^2}}{x} + C}$$



Ex5 $\int \frac{\sqrt{x^2 - 1}}{x^4} \, dx$ $x = \sec\theta$, $0 \le \theta \le \frac{\pi}{2}$; $dx = \sec\theta \tan\theta \, d\theta$

$$= \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^4\theta} \sec\theta \tan\theta \, d\theta$$

$$= \int \frac{\tan\theta}{\sec^4\theta} \sec\theta \tan\theta \, d\theta$$

$$= \int \cos^3\theta \tan^2\theta \, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \cos^3\theta \, d\theta$$

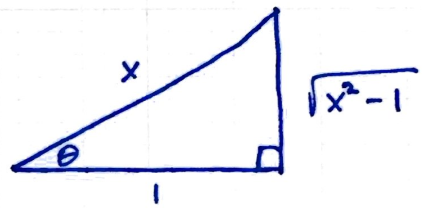
$$= \int \sin^2\theta \cos\theta \, d\theta$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \int u^2 \, du$$

$$= \frac{1}{3} u^3 + C$$



$$= \frac{1}{3} \sin^3\theta + C$$

$$\boxed{= \frac{1}{3} \cdot \frac{(x^2-1)^{3/2}}{x^3} + C}$$

Practice

$$\int \frac{x}{\sqrt{x^2+4}} \, dx$$

$u = x^2+4$

$du = 2x \, dx$

$$= \frac{1}{2} \int u^{-1/2} \, du$$

$$= u^{1/2} + C$$

$$= \boxed{\sqrt{x^2+4} + C.}$$