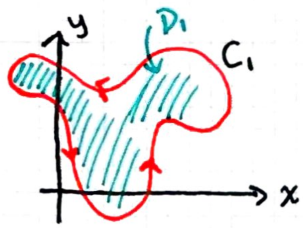
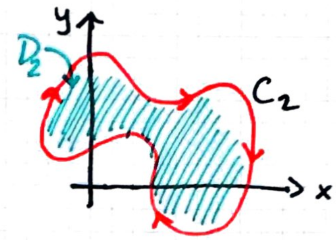
16.4 Green's Theorem

We say by convention that the positive orientation of a simple closed curve C is the single traversal of C in the counterclockwise direction.



positively oriented



negatively oriented

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let D be the region enclosed by C. If P and Q have continuous partial derivatives on an open region containing D, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Note We sometimes use the notation $\oint_C$ or $\oint_C$ to denote taking the integral over C using the positive orientation. We can also write ∂D to denote the positively oriented boundary curve of D. So Green's Thm states

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

Example 1 Evaluate the integral where C with center $(0,0)$, radius 2.

$$\oint_C (x-y)\,dx + (x+y)\,dy$$

by (a) directly from the definition and (b) using Green's Theorem.

Solution (a) We parametrize C by $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$. Then $dx = -2\sin t\,dt$ and $dy = 2\cos t\,dt$, and

$$\int_C (x-y)\,dx + (x+y)\,dy = \int_0^{2\pi} (2\cos t - 2\sin t)(-\sin t)\,dt + \int_0^{2\pi} (2\cos t + 2\sin t)(\cos t)\,dt$$

$$= \int_0^{2\pi} 4(\cos^2 t + \sin^2 t)\,dt = \int_0^{2\pi} 4\,dt = 8\pi.$$

(b) Using Green's Thm, $P = x-y \Rightarrow \frac{\partial P}{\partial y} = -1$; $Q = x+y \Rightarrow \frac{\partial Q}{\partial x} = 1$, so

$$\oint_C (x-y)\,dx + (x+y)\,dy = \iint_D (1-(-1))\,dA$$
$$= 2\iint_D dA$$
$$= 2\,A(D) = 2\cdot\pi(2)^2 = 8\pi.$$

Remark When the region D is simple enough, Green's Thm is often much more efficient for computing line integrals over simple closed curves.

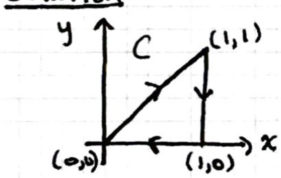
Recall that if $\vec{F} = \langle P, Q\rangle$, then $\int_C P\,dx + Q\,dy$ is the same as writing $\int_C \vec{F}\cdot d\vec{r}$.

Note also that if C has negative orientation, then $-C$ is positively oriented, so

$$\int_C P\,dx + Q\,dy = -\oint_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Example 2 Use Green's Thm to evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y) = \langle \sqrt{x^2+1}, \tan^{-1}(x)\rangle$ and C is the triangle from $(0,0)$ to $(1,1)$ to $(1,0)$ to $(0,0)$.

Solution



$$\int_C \vec{F}\cdot d\vec{r} = -\oint_C \vec{F}\cdot d\vec{r} \quad \text{since } C \text{ is negatively oriented.}$$

So
$$\int_C \vec{F}\cdot d\vec{r} = -\iint_D \left(\frac{\partial}{\partial x}(\tan^{-1}(x)) - \frac{\partial}{\partial y}(\sqrt{x^2+1})\right) dA$$
$$= -\iint_D \frac{1}{1+x^2}\,dA$$
$$= -\int_0^1\int_0^x \frac{1}{1+x^2}\,dy\,dx = -\int_0^1 \frac{x}{1+x^2}\,dx$$
$$= -\frac{1}{2}\int_1^2 \frac{du}{u} = \boxed{-\frac{1}{2}\log 2}$$

$u = 1+x^2$
$du = 2x\,dx$

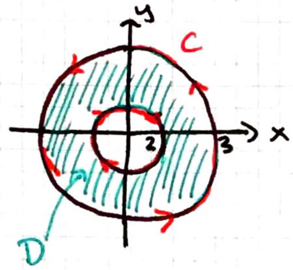
Example 3 Evaluate $\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy$, where C is the boundary of the region between the circles $x^2+y^2=4$ and $x^2+y^2=9$.

Remarks (1) So far we've only stated Green's Thm for simple regions, ie, where D is both type I and type II. But Green's Thm is still true whenever D is a finite union of nonoverlapping simple regions.

(2) We can further extend Green's Thm when D is not simply connected (as in Example 2). In this case, $C = \partial D$ is positively oriented if

D is always on the left as C is traversed.

Solution to Example 3



The region D is the annulus $\{(r,\theta) : 0 \le \theta \le 2\pi,\ 2 \le r \le 3\}$, so by the extended version of Green's Thm,

$$\oint_C (1-y^3)\,dx + (x^3 + e^{y^2})\,dy$$

$$= \iint_D (3x^2 + 3y^2)\,dA$$

$$= 3\int_0^{2\pi}\int_2^3 r^3\,dr\,d\theta$$

$$= 6\pi\int_2^3 r^3\,dr = \frac{195}{2}\pi.$$

Green's Theorem can also be used in the opposite direction if it happens to be easier to calculate the line integral. One useful application is for computing the area of D. Observe that if $P(x,y) = -\frac{1}{2}y$ and $Q(x,y) = \frac{1}{2}x$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. So,

$$A(D) = \iint_D 1\,dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Example 4 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution The ellipse is parametrized by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$. So by the above equation,

$$A = \frac{1}{2}\oint_C (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$