## 14.1 Functions of several variables

A function of two variables takes an ordered pair $(x,y)$ in a set $D$ and assigns a unique value $f(x,y)$. $D$ is the domain. The set of possible values is the range.

Often write $z=f(x,y)$.

**Example 1** Find and sketch the domain of the functions.

(a) $f(x,y)=\sqrt{x^2+y^2-4}$

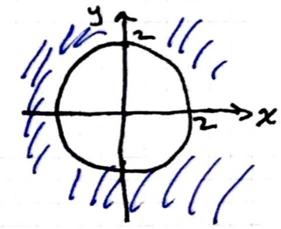
(b) $f(x,y)=\dfrac{\sqrt{y-x^2}}{1-x^2}$

(c) $f(x,y)=\log(x+1)+\log(y+1)$

(d) $f(x,y)=\log((x+1)(y+1))$

**Solution**

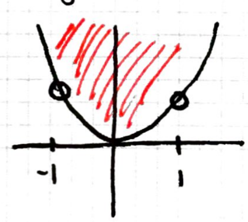
(a) Need $x^2+y^2-4\geq 0 \iff x^2+y^2\geq 4$. This is the region outside or on the circle of radius 2 centered at $(0,0)$.




(b) Need $y-x^2\geq 0$ and $1-x^2\neq 0$

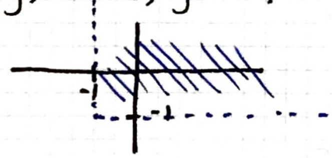
$y\geq x^2$ and $x\neq \pm 1$.

$D=\{(x,y): y\geq x^2,\ x\neq \pm 1\}$



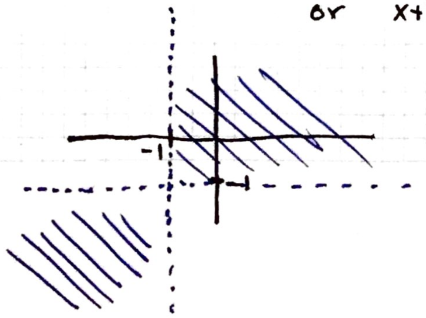

(c) Need $x+1>0 \iff x>-1$ and $y+1>0 \iff y>-1$

$D=\{(x,y): x>-1,\ y>-1\}$




(d) Need $(x+1)(y+1)>0 \iff x+1>0$ and $y+1>0$

or $x+1<0$ and $y+1<0$.



Notice $\log(x+1) + \log(y+1) = \log[(x+1)(y+1)]$ algebraically, but as functions they have different domains.

Example 2 Find the domain and range of

$$f(x,y,z) = \frac{1}{\sqrt{9-x^2-y^2-z^2}}.$$

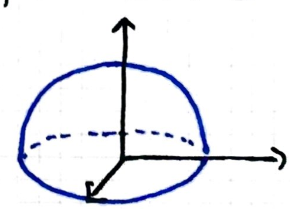
Solution We need $x^2+y^2+z^2 < 9$, so $D = \{(x,y,z) : x^2+y^2+z^2<9\}$. This is the inside of the sphere of radius 3, center $(0,0,0)$.

To find the range, note that the smallest $x^2+y^2+z^2$ can be is 0. So $0 \le x^2+y^2+z^2 < 9$. When $x^2+y^2+z^2=0$, $f(0,0,0) = \frac{1}{\sqrt{9}} = \frac{1}{3}$. And as $x^2+y^2+z^2$ approaches 9, $f(x,y,z)$ approaches $\infty$. So the range is $[\frac{1}{3}, \infty)$.

A graph of a function $f$ is $\{(x,y,z) \in \mathbb{R}^3 : z = f(x,y) \text{ and } (x,y) \in D\}$. (If $f$ is a function of two variables)

Example 3 Sketch the graph of $g(x,y) = \sqrt{9-x^2-y^2}$.

Solution squaring both sides, $x^2+y^2+z^2 = 9$, so, this is the upper hemisphere since $z = \sqrt{9-x^2-y^2} \ge 0$.



To help visualize 3D graphs, we often use level curves, or contours, which are curves with equations $f(x,y) = k$. It's basically a topographical map of the surface $z = f(x,y)$.

Example 4 Sketch a contour map of $f(x,y) = \log(x^2+4y^2)$.

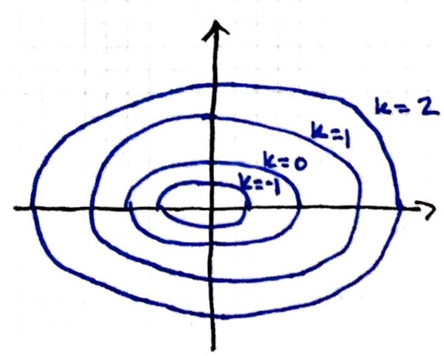
Solution Set $\log(x^2+4y^2) = k$

$$\iff x^2+4y^2 = e^k$$

But for constant $k$, $e^k$ is constant.

Equivalently, $$\frac{x^2}{e^k} + \frac{y^2}{e^k/4} = 1.$$

So these are ellipses with major axis $e^k$, minor axis $e^k/4$.



Example 5 For each of the functions, match its graph and contour map from the page that follows.

(a) $f(x,y) = \sin(xy)$

(b) $f(x,y) = \cos(4x^2 + y^2)$

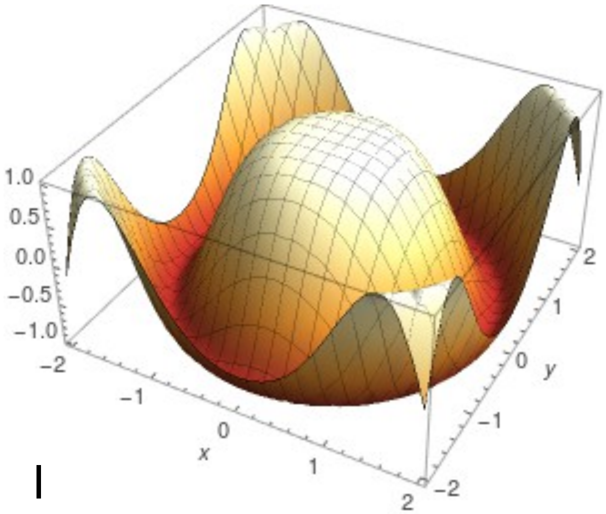
(c) $f(x,y) = \cos(x^2 + y^2)$.

Solution

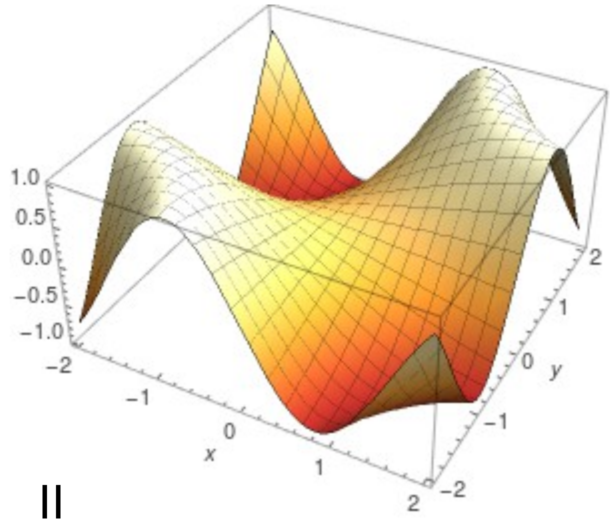
(a) level curves are hyperbolas $xy = \sin^{-1}(k)$ or the axes if $\sin^{-1}(k) = 0$. $\boxed{\text{A. and II.}}$

(b) $4x^2 + y^2 = \cos^{-1}(k)$ are ellipses for $\cos^{-1}(k) > 0$. $\boxed{\text{B. and III}}$

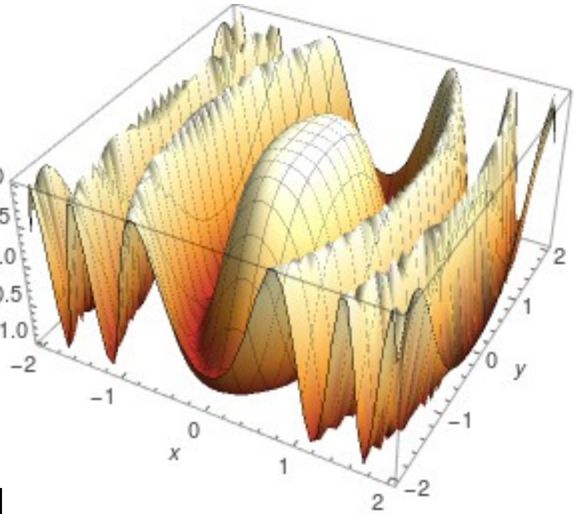
(c) $x^2 + y^2 = \cos^{-1}(k)$ are circles for $\cos^{-1}(k) > 0$. $\boxed{\text{C. and I}}$



I

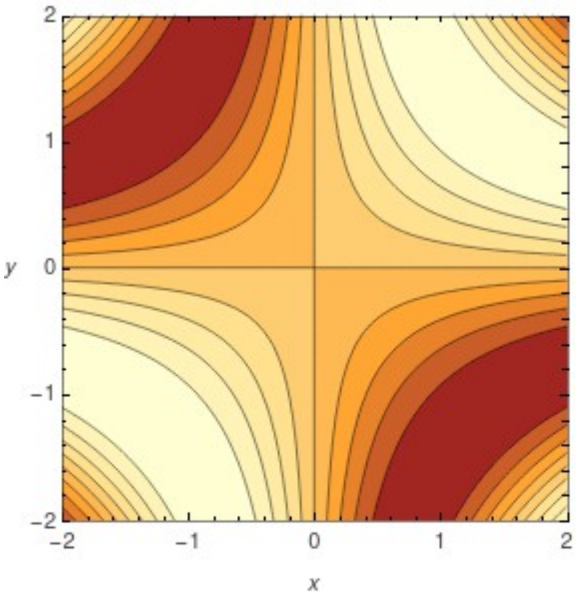


II

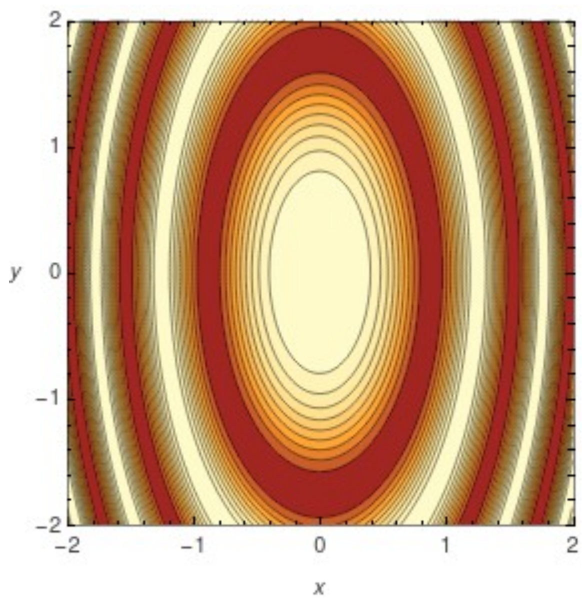


III

A



B



C