14.4 Tangent planes and linear approximations

In calc 1, we use the tangent line of a curve to approximate its values near some point. With $z = f(x,y)$ a surface, we can accomplish the same goal using the tangent plane.

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$, is

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).}$$

Notice This is similar to the equation of the tangent line from Calc I: $y - y_0 = f'(x_0)(x - x_0)$.

Example 1 Find the tangent plane to the surface $z = \frac{x}{y^2}$ at the point $(-4, 2, -1)$.

Solution First we compute $\frac{\partial z}{\partial x} = \frac{1}{y^2}$ and $\frac{\partial z}{\partial y} = \frac{-2x}{y^3}$.

At the specified point $f_x(-4, 2) = \frac{1}{4}$, $f_y(-4, 2) = \frac{8}{8} = 1$

So an equation for the tangent plane is

$$z + 1 = \frac{1}{4}(x + 4) + (y - 2). \quad \text{or} \quad z = \frac{1}{4}x + y - 2.$$

Linear approximations

If we zoom in on a graph of $z = x/y^2$ and its tangent plane at $(-4, 2, -1)$, the two surfaces will be very close together. Therefore, for (x,y) "close" to $(-4, 2)$, the plane

$$L(x,y) = \frac{1}{4}x + y - 2$$

is a good approximation for $f(x,y) = x/y^2$. The function L is called the linearization of f at $(-4, 2)$ and the approximation

$$f(x,y) \approx \frac{1}{4}x + y - 2$$

is called the linear approximation or tangent plane approximation of f at $(-4, 2)$.

In general, the linearization of f at (a,b) is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

and the linear approximation of f at (a,b) is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Example 2 Approximate $f(1.02, .97)$, where

$$f(x,y) = 1 - xy\cos(\pi y)$$

using its linear approximation at $(1,1)$.

Solution $f_x(x,y) = -y\cos(\pi y)$, $f_y(x,y) = \pi xy\sin(\pi y) - x\cos(\pi y)$

So, $f_x(1,1) = f_y(1,1) = 1$. Then since $f(1,1) = 2$,

$$f(x,y) \approx 2 + (x-1) + (y-1)$$
$$= x+y$$

So $f(1.02, .97) \approx 1.99$. Actual value: $1.9850090\ldots$, so we are pretty close.

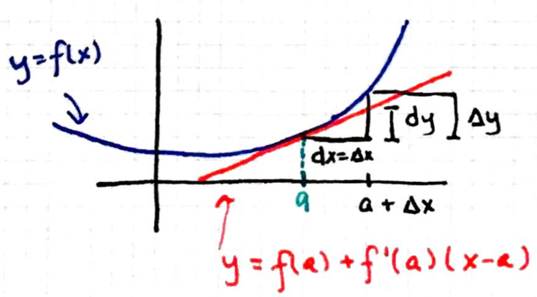
A function f of two variables is differentiable at a point (a,b) if its linear approximation is "good" when (x,y) is near (a,b). [What exactly "good" means is in Definition 7.]

It turns out, if f_x and f_y exist and are continuous, then the function $z = f(x,y)$ is differentiable. [This is all done point by point as with continuity.]

Differentials

In one-variable, given $y = f(x)$, the differential of y is

$$dy = f'(x)\,dx.$$



We set $dx = \Delta x$, then dy is the change in the height of the tangent line and Δy is the change in height of f.

In two variables, $z=f(x,y)$, now dx and dy are independent variables, and the differential dz, also called the total differential, is

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

For a picture, see figure 7 on page 932.

Differentials allow us to estimate changes or errors in measurement of differentiable functions.

Example 3 The length and width of a rectangle are measured as 30 cm and 24 cm, with error $\leq .1$ cm each. Estimate the maximum error in the calculated area of the rectangle.

Solution The area of the rectangle is given by $A=xy$, so

$$dA = y\,dx + x\,dy.$$

We're given $|\Delta x| \leq .1$ and $|\Delta y| \leq .1$, so to estimate the maximum error in area, $|\Delta A|$, we take $dx=dy=.1$. We are also given $x=30$ and $y=24$. So

$$|\Delta A| \approx dA = (24)(.1) + (30)(.1) = 5.4.$$

Example 4 Estimate the amount of metal in a closed cylindrical can that is 10 cm high, 4 cm in diameter if the metal in the top and bottom is .1 cm thick and the metal in the sides is .05 cm.

Solution The volume is $V = \pi r^2 h$, and

$$\Delta V \approx dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$
$$= 2\pi r h\,dr + \pi r^2\,dh.$$

Here, $dr = .05$, $dh = .1+.1 = .2$, $h=10$, $r=2$, so

$$dV = 2\pi(20)(.05) + \pi(2)^2(.2) = 2.8\pi \approx 8.8 \text{ cm}^3.$$

14.5 The chain rule

There are several versions of the chain rule for functions of more than one variable.

Case 1 $z=f(x,y)$ is differentiable function of x and y, and $x=g(t)$, $y=h(t)$ are differentiable functions of t. Then z is a differentiable function of t, and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Alternatively, we could write

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Notice the similarity to the differential $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.

Example 5 If $z = x^2y + 3xy^4$, where $x = \sin 2t$, $y = \cos t$, find dz/dt.

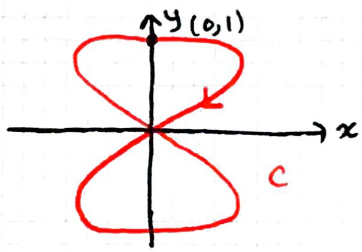
Solution The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t).$$

Remark It is not necessary to put every thing in terms of t.

To give the above example clearer context, observe that the following graph shows $\vec{r}(t) = \langle 2\sin t, \cos t\rangle$.

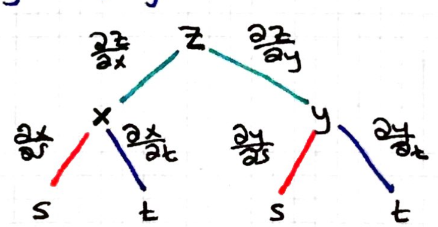


So the functions $x = \sin 2t$, $y = \cos t$ traces out the curve C, and $z = f(x,y)$ is a function on the curve. For example, t could be the temperature, so $z = f(x,y)$ gives the temperature of a point on the curve, and dz/dt gives the rate of change of temperature along C.

Case 2 $z = f(x,y)$ and $x = g(s,t)$, $y = h(s,t)$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Case 2 can be generalized to any number of variables. The following tree diagram may help you rembember the chain rule.



Example 6 Find $\frac{\partial z}{\partial s}$ where

$$z = \tan^{-1}(x^2+y^2), \quad x = s\log t, \quad y = te^s.$$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{2x}{1+(x^2+y^2)^2}\cdot \log t + \frac{2y}{1+(x^2+y^2)^2}\cdot te^s$$

Implicit Differentiation

Suppose the equation $F(x,y)=0$ defines y implicitly as a differentiable function of x. That is, $y=f(x)$, and $F(x,f(x))=0$ for all x in domain of f. If F is differentiable,

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

and if $\partial F/\partial y \neq 0$, we get

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y}$$

(since $dx/dx = 1$).

Note This is related to the Implicit Function Theorem.

Example 7 Find dy/dx for $\cos(xy) = 1+\sin y$.

Solution Write $F(x,y) = 1+\sin y - \cos(xy)$. Then $F(x,y)=0$, and

$$F_x = y\sin(xy), \quad F_y = \cos y + x\sin(xy), \quad \text{so}$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-y\sin(xy)}{\cos y + x\sin(xy)}.$$