Instructions. Show all work, with clear logical steps. No work or hard-to-follow work will lose points.

Problem 1. (10 points) Let $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2 \rangle$. Find the domain and the equation of the tangent line to this curve at $(0, 2, 1)$. (Hint: First find a tangent vector at the given point.)

Solution. The domain of $\ln t$ is $t > 0$, the domain of $2\sqrt{t}$ is $t \geq 0$ and the domain of t^2 is $\mathbb{R}$. So the domain of $\mathbf{r}$ is $(0, \infty)$.

Notice that $t = 1$ at the point $(0, 2, 1)$. Now $\mathbf{r}'(t) = \left\langle 1/t, 1/\sqrt{t}, 2t \right\rangle$, so a tangent vector at $t = 1$ is $\mathbf{v} = \langle 1, 1, 2 \rangle$. This vector is parallel to the tangent line, and $(0, 2, 1)$ is a point on that line, so an equation of the tangent line is $\mathbf{L}(t) = \langle 0, 2, 1 \rangle + t \langle 1, 1, 2 \rangle$. □

Problem 2. (10 points) Find the curvature of the curve $x = t$, $y = e^t$, $z = \sin t$ at the point $(0, 1, 0)$.

Solution. Notice that $t = 0$ at the point $(0, 1, 0)$. The vector equation is $\mathbf{r}(t) = \langle t, e^t, \sin t \rangle$ The easiest way to do this problem is to use the formula

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3}.$$

We have

$$\begin{aligned} \mathbf{r}'(t) &= \left\langle 1, e^t, \cos t \right\rangle \\ \mathbf{r}''(t) &= \left\langle 0, e^t, -\sin t \right\rangle \end{aligned}$$

So

$$\begin{aligned} \mathbf{r}'(0) &= \langle 1, 1, 1 \rangle \\ \mathbf{r}''(0) &= \langle 0, 1, 0 \rangle \end{aligned}$$

and $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -1, 1, 0 \rangle$. So

$$\kappa(0) = \frac{\sqrt{2}}{\sqrt{3}^3} = \frac{1}{3}\sqrt{\frac{2}{3}}.$$

You could also do this using the formula

$$\kappa(t) = \frac{|\mathbf{T}'(\mathbf{t})|}{|\mathbf{r}'(t)|}.$$

From above, we already know $|\mathbf{r}'(t)| = \sqrt{3}$. We need to find $|\mathbf{T}'(t)|$. Now

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\mathbf{r}(t)} \\ &= \frac{1}{\sqrt{1 + e^{2t} + \cos^2 t}} \left\langle 1, e^t, \cos t \right\rangle, \end{aligned}$$

so

$$\mathbf{T}'(t) = \frac{1}{\sqrt{1+e^{2t}+\cos^2 t}}\left\langle 0, e^t, -\sin t\right\rangle - \frac{2e^{2t} - 2\cos t \sin t}{(1+e^{2t}+\cos^2 t)^{3/2}}\left\langle 1, e^t, \cos t\right\rangle$$

Evaluating at $t = 0$,

$$\begin{aligned}
\mathbf{T}'(\mathbf{0}) &= \frac{1}{\sqrt{1+1+1}}\langle 0,1,0\rangle - \frac{2-0}{2(1+1+1)^{3/2}}\langle 1,1,1\rangle \\
&= \frac{1}{\sqrt{3}}\langle 0,1,0\rangle - \frac{1}{3\sqrt{3}}\langle 1,1,1\rangle \\
&= \left\langle -\frac{1}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}}\right\rangle.
\end{aligned}$$

So $|\mathbf{T}'(0)| = \sqrt{\frac{1}{27}+\frac{4}{27}+\frac{1}{27}} = \frac{\sqrt{2}}{3}$, and again we conclude that

$$\kappa(0) = \frac{\sqrt{2}/3}{\sqrt{3}} = \frac{1}{3}\sqrt{\frac{2}{3}}.$$

□