When the math fails
Side-channel attacks on ECDH

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Student Colloquium Talk

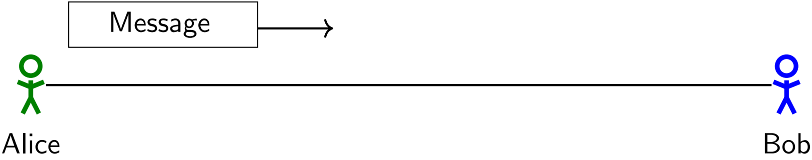
2 October 2019

Overview

1. Cryptography basics
   - The general problem
   - Classical Diffie-Hellman
2. Elliptic curve basics
   - What they look like
   - Group structure
   - Montgomery curves
3. ECDH
4. Side channel attacks
5. Bonus

The problem

The problem



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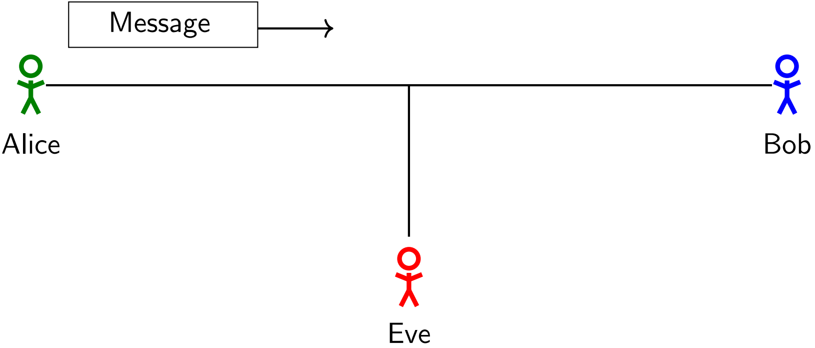

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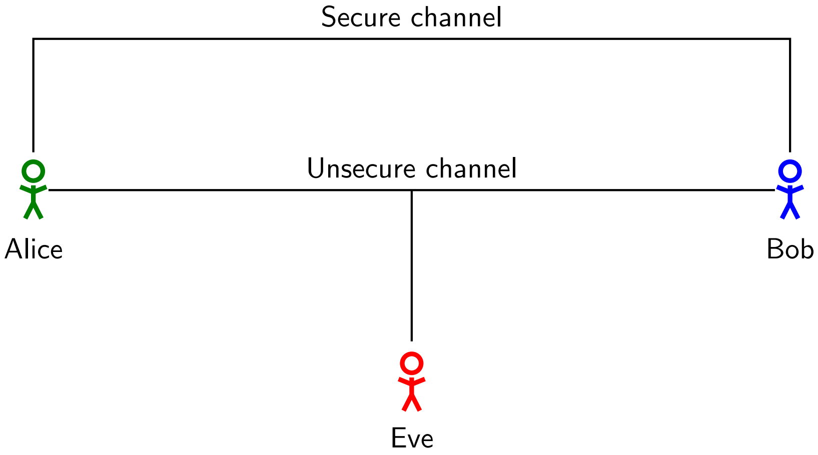

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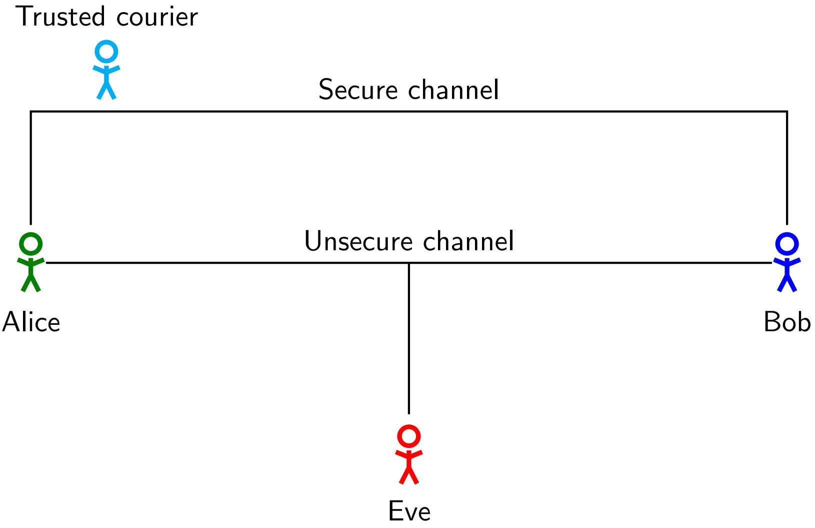
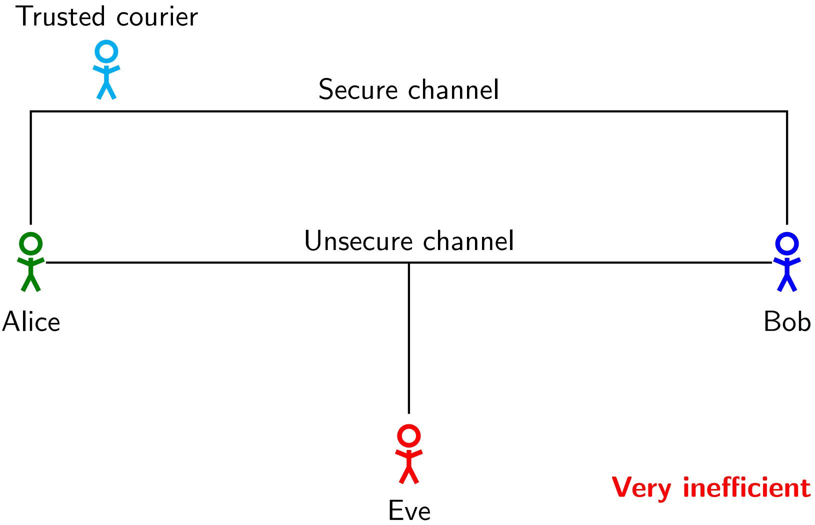

Trusted courier
Secure channel
Unsecure channel
Alice
Bob
Eve

The problem



Very inefficient

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- Asymmetric encryption is generally used to establish a shared key.

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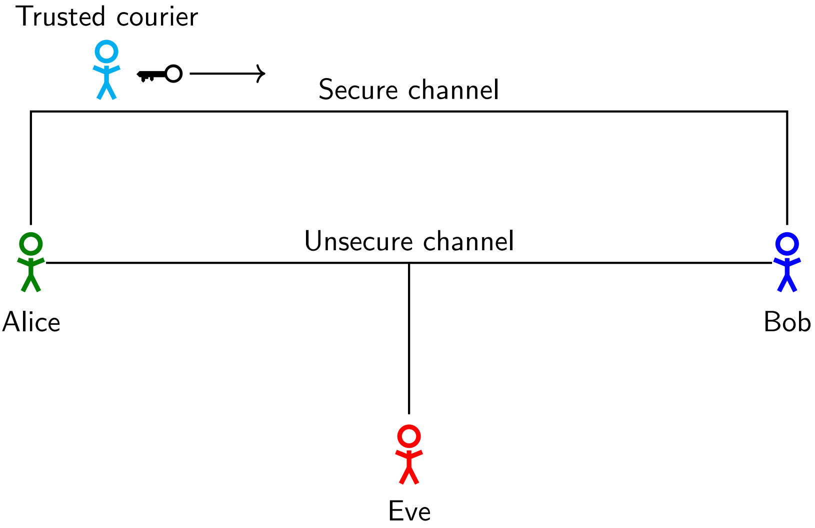

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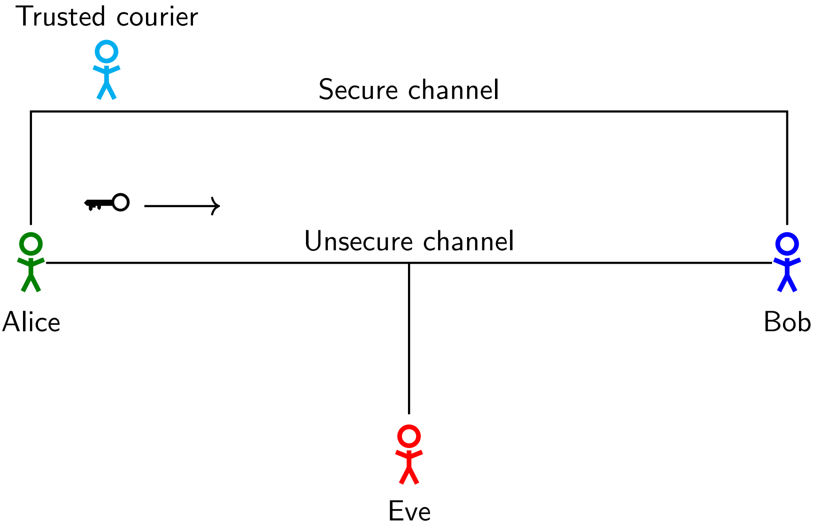

Discrete log problem (DLP)

Let p be a prime number, and let $a, b \in \mathbb{Z}$ such that $a, b \not\equiv 0 \bmod p$. Suppose we know there exists $k \in \mathbb{Z}$ such that

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More generally, if G is a group and $a, b \in G$, and given

$$a^k = b,$$

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Public parameters:
g, p

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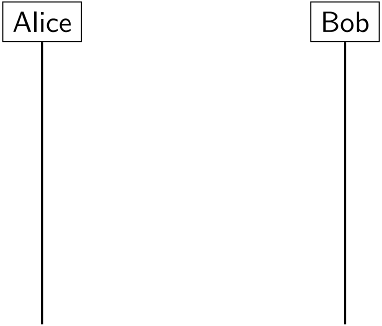

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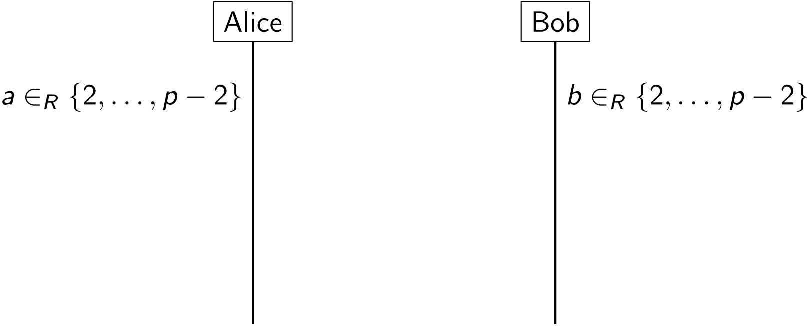

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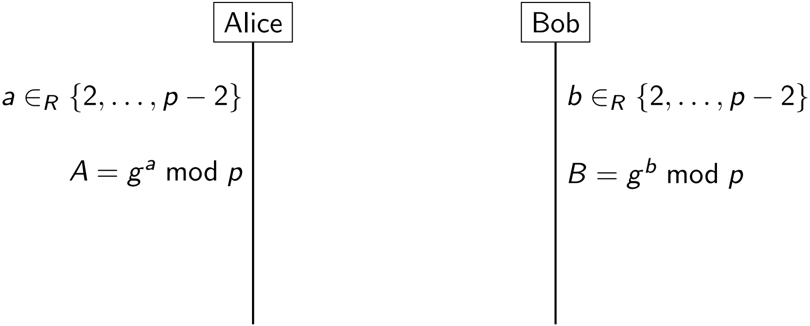

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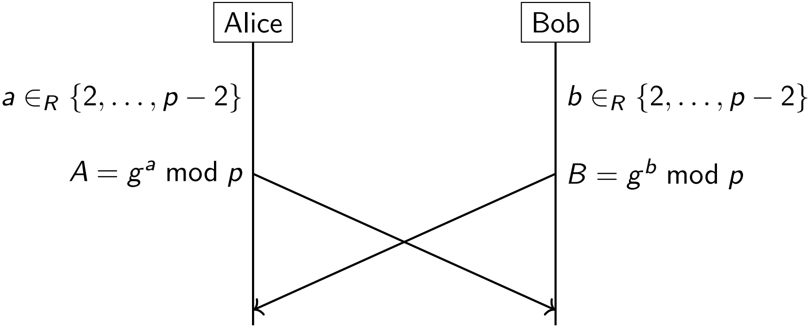

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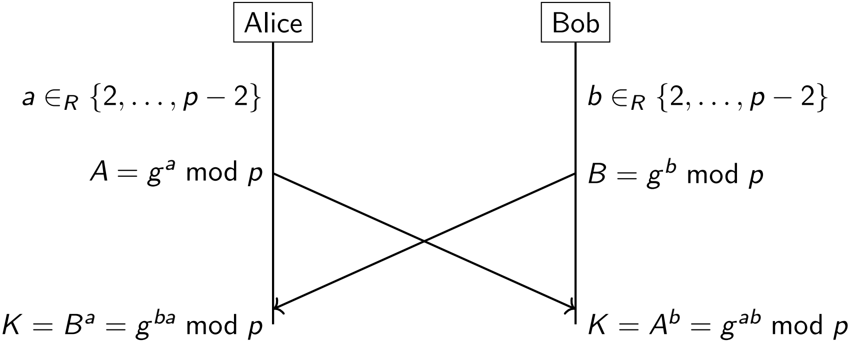
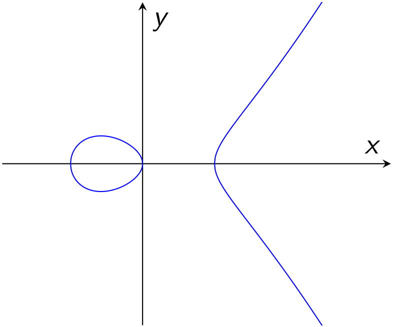


Elliptic curves



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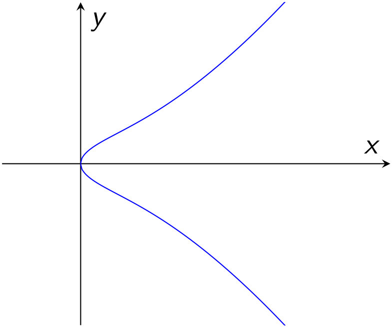

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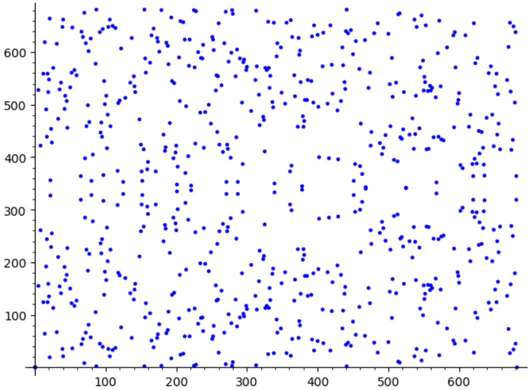
Elliptic curve modulo p



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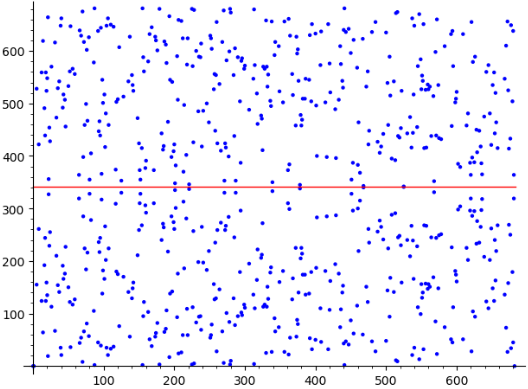

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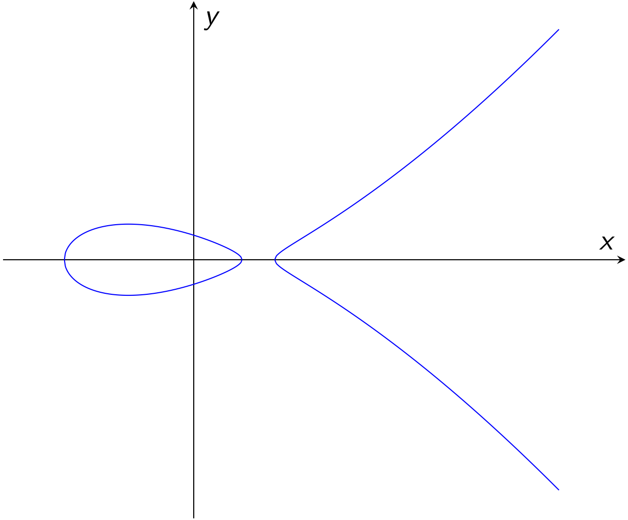

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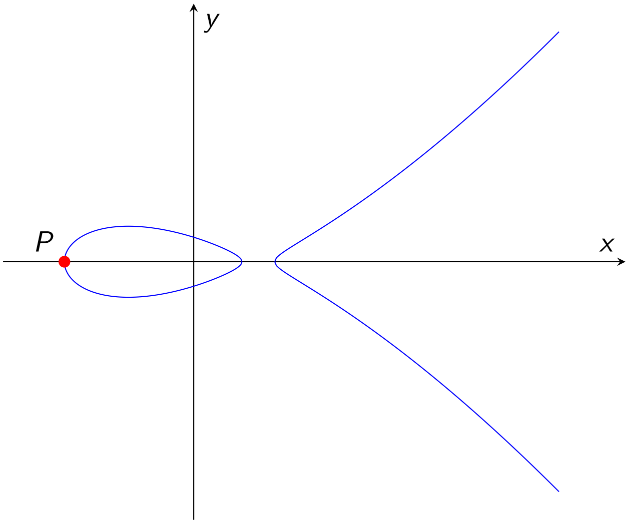

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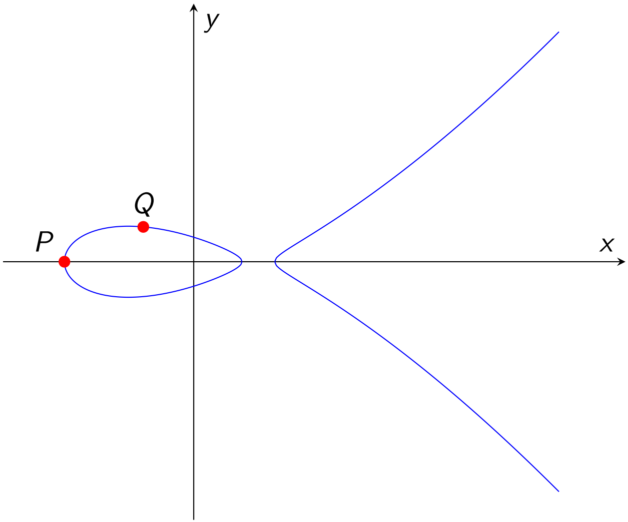

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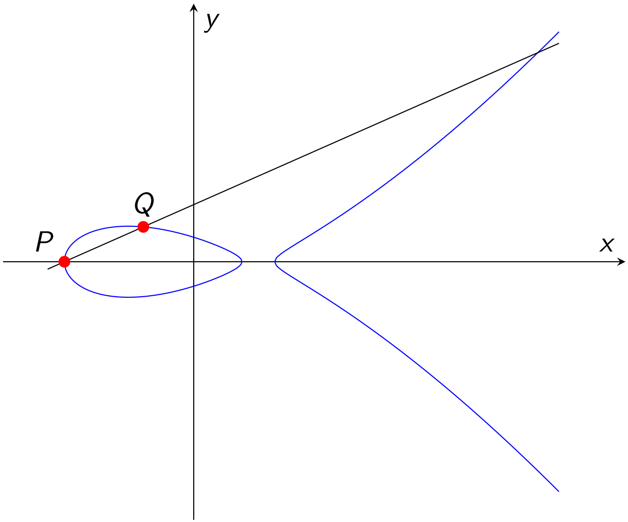

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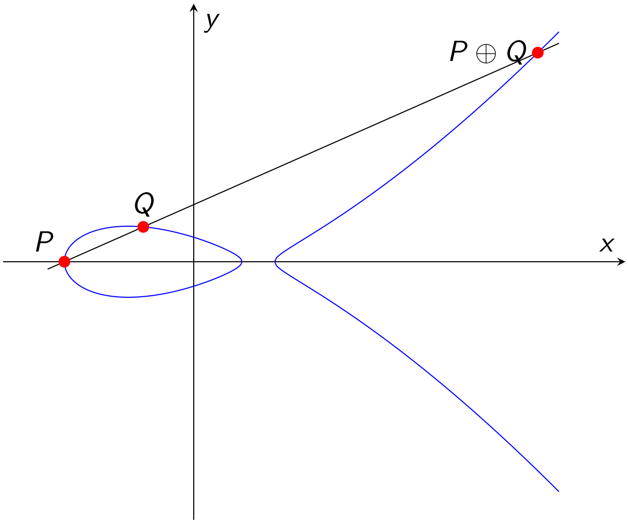

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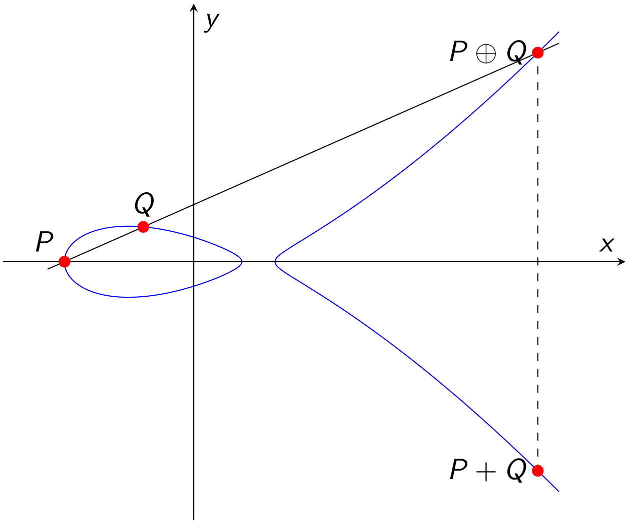

Elliptic curves

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- An elliptic curve $E/\mathbb{F}_q$ is a nonsingular curve satisfying the cubic equation
$$y^2 = x^3 + Ax + B.$$
- The set of points on E lying in $\mathbb{F}_q$ plus the point at infinity turns E into a group, denoted $E(\mathbb{F}_q)$.
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- For efficiency reasons, Montgomery curves are used in practice.
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- ECDH = "Elliptic Curve Diffie-Hellman"
- Alice and Bob agree on an elliptic curve E and a field $\mathbb{F}_q$ such that the DLP is hard for $E(\mathbb{F}_q)$.
- We require that $q + 1 - \#E(\mathbb{F}_q) \neq 0 \mod p$.
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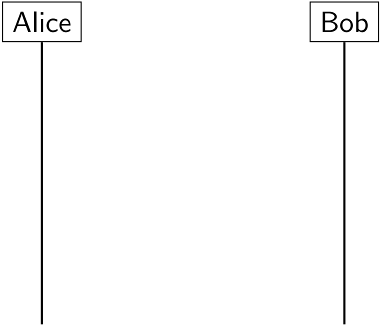

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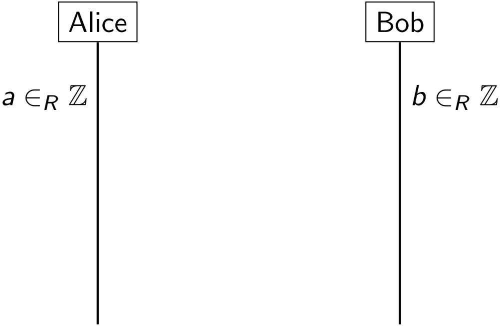

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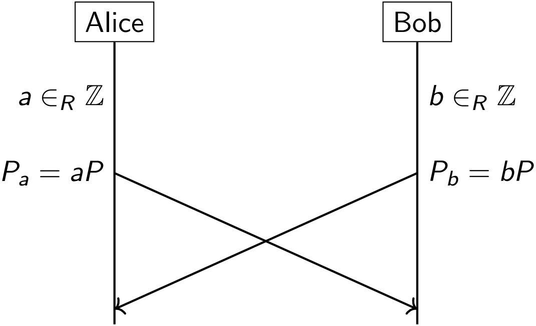

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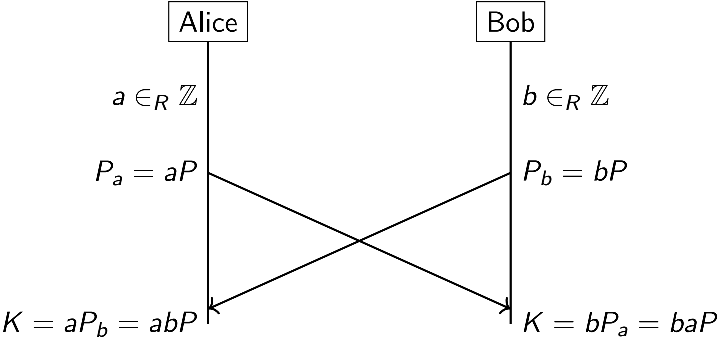

EC DLP

Let P, Q be points on $E(\mathbb{F}_q)$. Suppose we know there exists $d \in \mathbb{Z}$ such that

$$Q = dP.$$

The **elliptic curve discrete log problem** is to find d.

Advantages of ECDH

- Using elliptic curves allows for *much* smaller key sizes: an RSA 4096-bit key provides the same level of security as a 313-bit EC key.
- The group law for elliptic curves can be performed efficiently.
- No known subexponential algorithm to solve DLP in this setting.

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Side-channel analysis

Side-channel attack	An attack on the physical implementation of a cryptosystem
Timing attack	Watch CPU clock to measure how long it takes to perform various operations
Simple power analysis (SPA)	Monitor power consumption during cryptographic process
Differential power analysis (DPA)	Statistically analyze power consumption measurements from a cryptosystem

Scalar multiplication in E

Algorithm 1: double-and-add

Input: Binary integer $d = (d_{\ell-1}, \ldots, d_0)$ and a point $P \in E$.
Output: Point $Q = dP \in E$.

$Q \leftarrow P$
for i from $\ell - 2$ to 0 **do**
    $Q \leftarrow 2Q$
    if $d_i = 1$ **then**
        $Q \leftarrow Q + P$
return Q

Weaknesses in Algorithm 1

- Addition and doubling require different amounts of power and CPU time
- This makes Algorithm 1 vulnerable to timing attacks and SPA

SPA-resistant scalar multiplication in E

Algorithm 2: Montgomery ladder

Input: Binary integer $d = (d_{\ell-1}, \ldots, d_0)$ and a point $P \in E$.
Output: Point $R_0 = dP \in E$.

$R_0 \leftarrow P$
$R_1 \leftarrow 2P$
for i from $\ell - 2$ to 0 **do**
  if $d_i = 0$ **then**
    $R_0 \leftarrow R_0 + R_1$
    $R_1 \leftarrow 2R_1$
  else
    $R_1 \leftarrow R_0 + R_1$
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return R_0

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- Other countermeasures are needed.

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- On next iteration set $R \leftarrow (-1)^b 2R$ and $S \leftarrow (-1)^b 2S$ for a random bit b.

Randomized projective coordinates

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$$E\colon\ Y^2Z = X^3 + AX^2Z + BZ^3,$$

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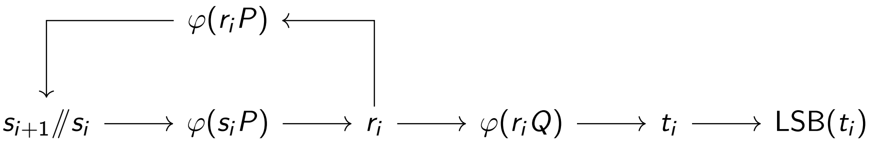
- In projective space $(X : Y : Z)$ and $(\lambda X : \lambda Y : \lambda Z)$ are equivalent points on E for any scalar λ.
- Given $P = (X : Y : Z)$, choose random integer λ and use $P' = (\lambda X : \lambda Y : \lambda Z)$.

Dual_EC_DRBG

- $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$
- $E\colon y^2 =$ $x^3 - 3x + 4105836372515214212932612978004726840911444101599\backslash\backslash$ $3725554835256314039467401291$
- P a generator of $E(\mathbb{F}_p)$
- $Q \in E(\mathbb{F}_p)$ a specified constant

How it works

- $\varphi\colon E(\mathbb{F}_p) \to \mathbb{Z}$ via $\varphi(x, y) = x$
- s_0 a randomly chosen seed
- $r_i = \varphi(s_i P)$, $t_i = \varphi(r_i Q)$, $s_{i+1} = \varphi(r_i P)$



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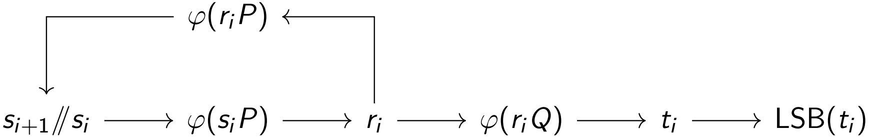

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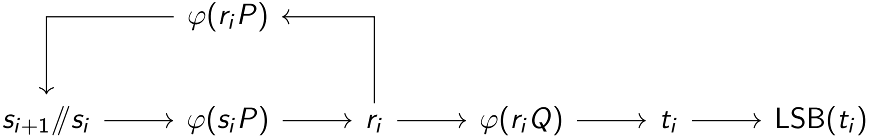

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- $\text{LSB}(t_i)$ only cuts off 16 bits of information.

The attack

$$r_i \longrightarrow \varphi(r_i P) \longrightarrow s_{i+1}$$

$$s_{i+1} /\!/ s_i \longrightarrow \varphi(s_i P) \longrightarrow r_i \longrightarrow \varphi(r_i Q) \longrightarrow t_i \longrightarrow \mathrm{LSB}(t_i)$$

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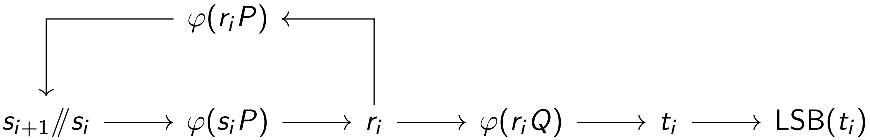

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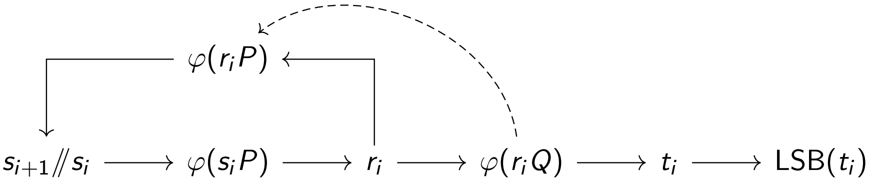

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- Given $\mathrm{LSB}(t_i)$, there are 2^{16} possibilities for t_i.
- Find which ones lie on E.
- This allows us to find r_iQ. But we want r_iP.

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- Then $\varphi(eA) = \varphi(er_i Q) = \varphi(r_i eQ) = \varphi(r_i P) = s_{i+1}$
- But we still have to solve the DLP to find e, so we're still safe?

Suspicious events

- When NIST published this standard, P and Q were predetermined.
- It was not published how Q was found.
- If an attacker knows $dP = Q$, he can easily compute e such that $eQ = P$.
- It was later revealed that the NSA chose P and Q, and the Snowden leaks suggest that they deliberately inserted a backdoor into this standard.

Questions?