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Sums of 3 squares

1. History & Results

Let K be a number field, $d = (K:\mathbb{Q})$
$\mathcal{O}$: ring of integers of K.

Let $V = K^n \supset L = \mathcal{O}^n$, a lattice

Fix $q(\bar{x}) = \sum a_{ij} x_i x_j$ $\qquad a_{ij} \in \mathcal{O}$

a non-degenerate integral quadratic form on V.

We are interested in the question of integral representabili

Q (IR) Given $\alpha \in \mathcal{O}$, when (and if so how often) is α integrally represented by (q, L), i.e., when can we solve

$$\alpha = q(\vec{l}) \qquad \text{with} \qquad \bar{l} \in L = \mathcal{O}^n \ ?$$

If we let

$$r_q(\alpha, L) = \left| \{ \vec{l} \in L \ : \ q(\bar{l}) = \alpha \} \right|$$

we want to understand these numbers.

$\underline{n=2}$: Hilbert : Class Field Theory of relative quad. extensions of number fields

$\underline{n \geq 3}$: Hilbert's 11^{th} problem.

Results

1. Siegel (1930's). Siegel first solved the related problem of representability of α by the genus $gen(L)$ of L.

Let $G = O_q$ be the orthogonal group of q.

Def (i) Two lattices $M_1, M_2 \subset V$ are equivalent, or lie in the same class, if $\exists\, g \in G(K)$: $gM_1 = M_2$

(ii) Two lattices $M_1, M_2 \subset V$ lie in the same genus if $\exists\, g \in G(\mathbb{A})$ s.th. $gM_1 = M_2$

$$\Updownarrow$$

$\forall$ places v of K, $\exists\, g_v \in G(K_v)$ s.th. $g_v M_{1,v} = M_{2,v}$
i.e. the lattices are locall equivalent at all places of K.

Siegel completely solved, by analytic methods, the problem of local integral representability (LIR):

Q (LIR) Given $\alpha \in \mathcal{O}$, when can we solve

$$\alpha = q(\vec{m}) \quad \text{with } \vec{m} \in M \quad \text{for some } M \in Gen(L) \ ?$$

or

$$\alpha = q(\vec{l}_v) \quad \text{w/ } \vec{l}_v \in L_v \quad \text{locally for all } v \ ?$$

We will briefly discuss this solution later.

2. Siegel (1950's) Siegel showed that if $n \geq 4$ and the quadratic ~~space~~ module (q, L) was indefinite at some archimedean place then

- $\alpha \in \mathcal{O}$ was represented by some $M \in \mathrm{Gen}(L)$

$$\Updownarrow$$

$\alpha \in \mathcal{O}$ was represented by all $M \in \mathrm{Gen}(L)$

i.e. $L \; \mathbb{IR} \iff \mathbb{IR}$. This is an equidistribution result.

Hsia, Kneser : extended to $n = 3$ by alg. tech (1960,70's)

This reduces us to : • K totally real
• $q(\vec{x})$ (totally) positive definite.

3 Hsia-Kitaoka-Kneser (1970's) If $n \geq 5$, $\exists$ effective $C > 0$ s.th. if $\alpha \in \mathcal{O}$ with
- $\alpha \gg 0$ (totally positive)
- $N(\alpha) > C$

then α is rep. by $L \iff \alpha$ is rep by L_v for all v

- If $n = 4$, similar stmt for primitive reps.

4. Thm (C-PS-S, 2000) Let $q(\vec{x})$ be a pos. def. integral ternary quad form. Then $\exists$ an (ineffective) $C > 0$ s.th if $\alpha \in \mathcal{O}$ with
- α square free
- $\alpha \gg 0$
- $N(\alpha) > C$

Then α is represented by (q, L) iff α is represented locally by (q, L_v) for all places v of K.

<u>COR</u> Sum of 3 squares: $q(x) = x_1^2 + x_2^2 + x_3^2$ ✓

2. Theta Series

$(K$ totally real, $(K:\mathbb{Q}) = d)$

For simplicity, assume K has class # = 1 (strict ?)
We restrict to $n = 3$, $q(\vec{x})$ integral, pos definite, $L = \mathcal{O}^3 \subset V = K^3$.

Siegel's <u>analytic class invariant</u>: For $\tau \in \mathfrak{h}^d$,

$$\vartheta_q(\tau, L) = \sum_{\vec{\ell} \in L} e^{2\pi i \, \mathrm{tr}(q(\ell)\tau)}$$

$$= 1 + \sum_{\alpha \gg 0} r_q(\alpha, L) e^{2\pi i \, \mathrm{tr}(\alpha\tau)}$$

where $\mathrm{tr}(\alpha\tau) = \sum \alpha_i \tau_i$ w/ α_i the various embeddings of α in $\mathbb{R}$.

This is a Hilbert modular form of wgt $3/2$ for an appropriate congruence sgp $\Gamma = \Gamma_q \subset SL_2(\mathcal{O})$.

Siegel's analytic Genus invariant :

$$\vartheta_q(\tau, \mathrm{Gen}(L)) = \frac{\displaystyle\sum_{[M]\in \mathrm{Gen}(L)} \frac{\vartheta_q(\tau, M)}{o(M)}}{\displaystyle\sum_{[M]} \frac{1}{o(M)}} \in M_{3/2}(\Gamma)$$

$o(M) = \left|\{ g \in G(K) : gM = M \}\right|$. "units of M"

$$\vartheta_q(\tau, \mathrm{Gen}(L)) = 1 + \sum_{\alpha \gg 0} r_q(\alpha, \mathrm{Gen}(L))\, e^{2\pi i \operatorname{tr}(\alpha\tau)}$$

Siegel's solution to LIR:

- $\vartheta_q(\tau, \mathrm{Gen}(L))$ is an Eisenstein series
- $r_q(\alpha, \mathrm{Gen}(L)) = \overset{?}{c} \prod_v r_q(\alpha, L_v)$ (solving congruences) product of local densities
- $r_q(\alpha, \mathrm{Gen}(L)) \approx N(\alpha)^{\frac{1}{2}} L(1, \chi_\alpha) \gg N(\alpha)^{\frac{1}{2}-\varepsilon}$ if non-zero.
  $(n=3)$
  ineffective, Brauer-Siegel

There is a similar invariant attached to the spinor genus of L:

$$\mathrm{Spn}(L) = \{ [M] : M \in \mathrm{Spin}_q(\mathbb{A}) \cdot L \}$$

$$\vartheta_q(\tau, \mathrm{Spn}(L)) = \ldots$$

<u>Siegel</u>: $\vartheta_q(\tau, L) - \vartheta_q(\tau, Spn(L)) \in S_{3/2}(\Gamma)$ cusp forms.
$\vartheta_q(\tau, Spn(L)) - \vartheta_q(\tau, Gen(L)) \in S_{3/2}(\Gamma)$

If we let

$\hat{S}_{3/2}(\Gamma) = \langle \Theta\text{-series attached to 1-dim'l quad. forms} \rangle$

$\overset{\circ}{S}_{3/2}(\Gamma) = \hat{S}_{3/2}(\Gamma)^{\perp}$

Then <u>Schulze-Pillot</u> $\Rightarrow$ $\vartheta_q(\tau, L) - \vartheta_q(\tau, Spn(L)) \in \overset{\circ}{S}_{3/2}(\Gamma)$

$\vartheta_q(\tau, Spn(L)) - \vartheta_q(\tau, Gen(L)) \in \hat{S}_{3/2}(\Gamma)$

<u>Prop</u> (SP) The above forms in $\hat{S}_{3/2}(\Gamma)$ have large Fourier coeff, but they are distributed among only a finite # of explicitly determined square classes.

<u>Cor</u> For square free α, outside an explicit finite set,
$r_q(\alpha, Spn(L)) = r_q(\alpha, Gen(L))$.

So $r_q(\alpha, L) - r_q(\alpha, Gen(L)) = \tilde{a}(\alpha)$ is the Fourier

coefficient of some

$$\tilde{f}(\tau) = \sum_{\alpha \gg 0} \tilde{a}(\alpha) e^{2\pi i \operatorname{tr}(\alpha\tau)} \in \overset{\circ}{S}_{3/2}(\Gamma).$$

and we expect these to be small, ie, satisfy a Ramanujan bound.

<u>Recall</u>: $\quad r_q(\alpha, \mathrm{Gen}(L)) \gg N(\alpha)^{\frac{1}{2}-\varepsilon} \quad$ all $\varepsilon > 0$,
so to see that

$$r_q(\alpha, L) \sim r_q(\alpha, \mathrm{Gen}(L)),$$

which is the thrust of our theorem, we need an estimate

$$|\tilde{a}(\alpha)| \ll N(\alpha)^{\frac{1}{2}-\delta} \quad \text{for some } \underline{\text{fixed } \delta > 0}$$

(~~say~~ non-triv est. for Ramanujan)

3. Fourier coefficients & L-fcts

Another characterization of $\overset{\circ}{S}_{3/2}(\Gamma)$ is in terms of the Shimura correspondence

$$\Theta : S_{3/2}(\Gamma) \longrightarrow M_2(\Gamma') \qquad \Gamma' \subset SL_2(\mathcal{O})$$

$$\overset{\circ}{S}_{3/2}(\Gamma) = \left\{ \tilde{f} \in S_{3/2}(\Gamma) \ : \ \Theta(\tilde{f}) = \varphi \text{ is cuspidal} \right\}$$

Let $\varphi = \Theta(\tilde{f})$ for our $\tilde{f}$ above.

Then the theorem of Waldspurger, generalized ~~by~~ to totally real K by Shimura + Baruch-Mao, relates

$$\tilde{a}(\alpha) \longleftrightarrow L(\tfrac{1}{2}, \varphi, \chi_\alpha).$$

If we write $\displaystyle \varphi(\tau) = \sum_{\mu \gg 0} a(\mu)\, e^{2\pi i \operatorname{tr}\left(\frac{1}{d_K}\mu\tau\right)}$

w/ d_K the different of K and let $\displaystyle \lambda(\mu) = \frac{a(\mu)}{N(\mu)^{1/2}}$

The normalized L-fct of φ is then

$$L(s,\varphi) = \sum_{\substack{\mu \gg 0 \\ \text{mod } U_+}} \lambda(\mu) N(\mu)^{-s}.$$

For $\alpha \in \mathcal{O}$ square free we also have associated the ray class character χ_α assoc to $K(\sqrt{\alpha})/K$, and

$$L(s,\varphi,\chi_\alpha) = \sum_{\substack{\mu \gg 0 \\ \text{mod } U_+}} \frac{\chi_\alpha(\mu)\lambda(\mu)}{N(\mu)^s}$$

Then the result of Waldspurger, via Baruch-Mao, gives

<u>Prop</u> $\quad |\tilde{a}(\alpha)|^2 \ll N(\alpha)^{\frac{1}{2}} L(\tfrac{1}{2},\varphi,\chi_\alpha).$

So we now need to know $L(\tfrac{1}{2},\varphi,\chi_\alpha) \ll N(\alpha)^{\frac{1}{2}-2\delta}$.

The trivial bound for $L(s,\varphi,\chi_\alpha)$ on the $\frac{1}{2}$-line is the so-called "convexity" bound coming from the Functional equation for $L(s,\varphi,\chi_\alpha)$ and an "application of Phragmen-Lindelöf". It is

$$L(\tfrac{1}{2},\varphi,\chi_\alpha) \ll_\varepsilon N(\alpha)^{\frac{1}{2}+\varepsilon}.$$

So to solve Hilbert's 11^{th} problem — and get a bound towards Ramanujan for $S^0_{3/2}(\Gamma)$ — we need <u>any</u> sub-convex estimate.

Thm (C-PS-S, 2000)

$$L(\tfrac{1}{2}, \varphi, \chi_a) \ll N(a)^{\frac{1}{2} - \frac{7}{130} + \varepsilon}$$

Rmk The proof uses standard spectral methods, but adapted to the number field situation:

- approx. F.E.
- averaging over families of numerical characters
- Ramanujan bds for F.C. of holom Hilb. mod. forms
- $\frac{1}{9}$ bounds of Kim-Shahidi for ~~Hilb. modular~~ F.C. of Hilb. modular Maaß forms (at ∞ also)

Recently, using softer adelic ergodic techniques, Venkatesh has proven a weaker, but still sufficient, sub-convex bd

Thm (Venkatesh, 2005)

$$L(\tfrac{1}{2}, \varphi, \chi_a) \ll N(a)^{\frac{1}{2} - \frac{1}{24} + \varepsilon}$$

which is in fact valid over any number field.