ON THE SOLUTIONS OF A FAMILY OF QUARTIC THUE EQUATIONS II

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1. Diophatine equations

Definitions

(1) **A diophantine equation** is an equation of the form

$$P(x_1, \dots, x_d) = 0, \quad d \in \mathbb{N}^*$$

(2) If $P(a_1, \dots, a_d) = 0$ where $a_i \in \mathbb{Z}$ for $i = 1, \dots, d$, then $(a_1, \dots, a_d)$ is a **solution**.

(3) **A Thue equation** is an equation of the form

$$F(x, y) = a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_1 x y^{d-1} + a_0 y^d = A,$$

where $a_i, A \in \mathbb{Z}$ with $A \neq 0$.

The history of Diophantine equations is very rich and goes back to the antiquity. But Thue was first to study an infinite parametrized family of Thue equations. He proved that

$$(a+1)X^n - aY^n = 1, \quad x > 0,\ Y > 0$$

has only the solution $x = y = 1$ for a suitably large in relation to prime $n \geq 3$.

There are many methods to solve families of Thue equations:

(1) Baker's method,

(2) The hypergeometric method.

2. **Baker's method**

Let us consider a family of parameterized Thue equations

$$F(x, y) = A$$

(with a parameter n).

(1) Determine the obvious solutions of the equation.

(2) Define the number field $\mathbb{K} = \mathbb{Q}(\theta)$, where θ is a root of the polynomial $F(x, 1)$.

(3) Determine a system of independent units $\{\varepsilon_1, \dots, \varepsilon_r\}$ of $\mathbb{K}$. Therefore, there exist integers $u_1, \dots, u_r$ such that

$$(x - \theta y)^I = \varepsilon_1^{u_1} \dots \varepsilon_r^{u_r}$$

where r is the rank of $\mathbb{K}$.

(4) Estimate the regulator R of $\mathbb{K}$.

(5) Find an upper bound for $U = \max\{u_1, \dots, u_r\}$, in terms of R and $\log |y|$.

(6) Compute an upper bound of the linear form Λ in logarithms obtained by a Siegel's formula.

(7) Combine the results of steps 4, 5, 6 to find an upper bound for Λ in terms of U.

(8) Determine a lower bound for U.

(9) Combine the results of steps 7 and 8 to obtain a negative upper bound for $\log|\Lambda|$.

(10) Determine a negative lower bound for $\log|\Lambda|$.

(11) Use the lower and upper bounds obtained from 9 and 10 for $\log|\Lambda|$ to compute a numerical upper bound for the parameter n, say $n \in N$.

3. **A survey**

A list of families of Thue equations studied since 1990 can be obtained at:

http://www.opt.math.tu-graz.ac.at/ cheub/thue.html

Families of degree 3

The first family of Thue equations is solved by Emery Thomas in 1990:

$$x^3 - (a-1)x^2y - (a+2)xy^2 - y^3 = 1$$

The solutions are:

$$(0,-1),\ (1,0),\ (-1,1).$$

Other cubic families of Thue equations are solved by Thomas, Mignotte-Tzanakis, Wakabayashi, and T.

$$x^3 - (n^3 - 2n^2 + 3n - 31)x^2y - n^2xy^2 - y^3 = \pm 1,$$

The solutions are:

$$(0,\pm 1),\ (\pm 1, 0).$$

Families of degree 4

The first quartic family of Thue equations is solved by Attila Pethő in 1991:

$$x^4 - ax^3y - x^2y^2 + axy^3 + y^4 = \pm 1;$$

the solutions are:

$$\pm\{(0,1),\ (1,0),\ (1,1),\ (1,-1),\ (a,1),\ (1,-a)\}.$$

Many quartic families are solved by: Pethő, Mignotte-Pethő-Roth, Lettl-Pethő, Chen-Voutier, Wakabayashi, Heuberger-Pethő-Tichy, Dujella-Jadrijevič, and T.

$$x^4 - n^2x^3y - (n^3 + 2n^2 + 4n + 2)x^2y^2 - n^2xy^3 + y^4 = 1;$$

for $n \leq 5 \cdot 10^6$ and $n \geq 1.191 \cdot 10^{19}$ with n, $n+2$, n^2+4 squarefree. The solutions are:

$$\pm\{(0,1),\ (1,0)\}.$$

Families of degree 5

The first quintic family of Thue equations is solved by Clemens Heuberger in 1998:

$$x(x^2 - y^2)(x^2 - a^2y^2) - y^5 = \pm 1,$$

for $a > 3.6 \cdot 10^{19}$. The solutions are
$\{(0, \pm 1), (\pm 1, 0), (1, \pm 1), (-1, \pm 1), (a, \pm 1), (-a, \pm 1)\}$.

Two other families are solved by Gaál-Lettl, Levesque-Mignotte.

Family of degree 6

Only one sextic family of Thue equations is solved by Lettl-Pethő-Voutier in 1999:

$$x^6 - 2ax^5y - (5a+15)x^4y^2 - 20x^3y^3 + 5ax^2y^4 + (2a+6)xy^5 + y^6 \in \{\pm 1,\ \pm 27\},$$

with $-\frac{y}{2} \le x \le y$. The solutions are

$$(0,1),\ (1,1).$$

They used the hypergeometric method.

Family of degree 8

Only one octic family of Thue equations is solved by Heuberger-T-Ziegler in 2004:

$$X^8 - 8nX^7Y - 28X^6Y^2 + 56nX^5Y^3 + 70X^4Y^4 - 56nX^3Y^5 - 28nX^2Y^6 + 8nXY^7 + Y^8 = \pm 1$$

has only trivial solutions

$$(\pm 1, 0),\ (0, \pm 1)$$

for $n \geq 6.71 \cdot 10^{32}$, where

$$n \in \{a \in \mathbb{Z} : a + b\sqrt{2} = (1+\sqrt{2})^{2k+1} : k \in \mathbb{N}\}.$$

Recently, Heuberger-Tethö-Tychy and Ziegler studied some families of relative Thue equations.

4. **The new result**

Theorem 4.1. [**Togbé, 2000**]
For $n \geq 1.191 \cdot 10^{19}$ *and* $n \leq 5 \cdot 10^6$ *with* n, $n+2$, n^2+4 *squarefree. the equation*

(4.1)
$$\Phi_n(x,y) = x^4 - n^2x^3y - (n^3+2n^2+4n+2)x^2y^2 - n^2xy^3 + y^4 = 1.$$

has only the trivial solutions

$$\pm\{(1,0),\ (0,1)\}. \tag{4.2}$$

For this, we used two methods:

- a method proposed by M. Mignotte
- the Bilu-Hanrot's method.

Moreover, we have conjectured that (4.1) has only the solutions (4.4) for any parameter $n \geq 1$.

Here is the new result:

Theorem 4.2. [**Togbé, 2006**]
For $n \geq 1$, the family of parameterized Thue equations

$$\Phi_n(x,y) = x^4 - n^2x^3y - (n^3+2n^2+4n+2)x^2y^2 - n^2xy^3 + y^4 = 1 \tag{4.3}$$

has only the integral solutions

$$\pm\{(1,0),\ (0,1)\}; \tag{4.4}$$

except for $n = 1$ where we have

$$\pm\{(1,-1),\ (1,0),\ (0,1)\}. \tag{4.5}$$

5. **Sketch of the proof of Theorem 4.2**

First Step

Lemma 5.1. *Let us consider* $\mathcal{O} = \mathbb{Z}[\alpha, \beta, \varepsilon]$, *and* $\langle -1, \alpha, \beta, \varepsilon\rangle$ *a subgroup of the unit group.*
For $n \geq 33$, *we have*

$$I := [\mathcal{O}^\times : \langle -1, \alpha, \beta, \varepsilon\rangle] < 49.27 \log^3(n). \tag{5.1}$$

Proof. Let R be the regulator of $\langle -1, \alpha, \beta, \varepsilon\rangle$:

$$R = 2\log(\varepsilon)\left(\log^2(\alpha) + \log^2(\beta)\right). \tag{5.2}$$

Using asymptotic expressions, one can check that

$$10 \log^3(n) \leq R \leq 10.11 \log^3(n), \tag{5.3}$$

for $n \geq 33$. So $R > 0$ and α, β, ε are independent units. We find the following bound for the index

$$I = [\mathcal{O}^\times : \langle -1, \alpha, \beta, \varepsilon\rangle] \leq \frac{R}{\mathrm{Reg}\ \mathbb{Z}_{\mathbb{K}_n}} \leq \frac{10.11 \log^3(n)}{0.2052}. \tag{5.4}$$

So we obtain

$$I \leq 49.27 \log^3(n).$$

□

Lemma 5.2. *If (x, y) is a solution of type j, then (y, x) is a solution of type $j + 2$.*

Proof. We obtain $\left|\frac{x}{y} - \alpha_i\right| < \frac{1}{2y^2}$. This means that $\frac{x}{y}$ is a convergent to α_j.

Also we have

$$\left|x\right| \leq \left|x - \alpha_j y\right| + \left|\alpha_j y\right| \leq \left|\alpha_j y\right| + \frac{1}{y^2} \leq 2\left|\alpha_j y\right|.$$

Therefore we obtain

$$\left|\frac{1}{\alpha_j} - \frac{x}{y}\right| = \frac{|x - \alpha_j y|}{\left|\alpha_j x\right|} \leq \frac{1}{2\left|\alpha_j x y\right|} < \frac{1}{\left|xy\right|} \leq \frac{1}{4}. \tag{5.5}$$

This means that

$$\left|\frac{x}{\alpha_j} - y\right| = \min_{1\leq i\leq 4}\{\left|x - \alpha_i y\right|\}.$$

As

$$\alpha_3 = \frac{1}{\alpha_1}, \quad \alpha_4 = \frac{1}{\alpha_2}, \text{ i.e. } \frac{1}{\alpha_i} = \alpha_{i+2} \tag{5.6}$$

therefore we study the cases $j = 1$ and $j = 4$. □

One can check that $\gamma_i := x - \alpha_i y$ are units in $\mathcal{O}^{\times}$.

Lemma 5.3. *Let $n \geq 1$ and (x, y) be a solution to (4.1) of type j such that $y \geq 2$. Then*

$$|\gamma_j| \leq k_j \frac{1}{y^3}, \qquad \text{where } k_j = \begin{cases} \frac{8}{n^6} & \text{if } j = 1, \\ \frac{8}{n^2} & \text{if } j = 4, \end{cases} \tag{5.7}$$

$$\log|\gamma_i| = \log(y) + \log|\alpha_i - \alpha_j| + \begin{cases} \frac{1}{2n^8} & \text{if } j = 1, \\ \frac{1}{2n^4} & \text{if } j = 4. \end{cases} \tag{5.8}$$

Proof. For $i \neq j$ we have

$$y|\alpha_i - \alpha_j| \leq 2|\gamma_i|,$$

then

$$|\gamma_j| = \frac{1}{\prod_{i\neq j} |\gamma_i|} \leq \frac{8}{y^3} \cdot \frac{1}{\prod_{i\neq j} |\alpha_i - \alpha_j|}. \tag{5.9}$$

Since

$$\prod_{i\neq 1} |\alpha_i - \alpha_j| \geq \begin{cases} n^6 & \text{if } j = 1, \\ n^2 & \text{if } j = 4, \end{cases}$$

we obtain (5.7). Moreover, we know that

$$\frac{|\gamma_i|}{y\,|\alpha_i - \alpha_j|} = \left|1 + \frac{\gamma_j}{y(\alpha_i - \alpha_j)}\right|, \tag{5.10}$$

then taking the log of the previous expression and using expressions of α_i, (5.7) we obtain (5.8). □

Lemma 5.4. *Let* (x, y) *be a solution to* (4.1) *with* $y \geq 2$ *and* $n \geq 33$. *Then*

$$\log y \geq 2.485 n \log(n). \tag{5.11}$$

Proof. If (x, y) is a solution to (4.1), then we determine asymptotic expressions of u_2, u_3.

For each j, we define the following linear combinations of the u_k such that we eliminate, if it is possible, the terms in $\log(n)\log(y)$, $\log^2(n)$, and $\log(n)$ appearing in the expressions of the $\frac{Ru_2}{I}$ and $\frac{Ru_3}{I}$:

$$\frac{Rv_j}{I} = \begin{cases} \frac{Ru_2}{I} - \frac{2Ru_3}{I} = \left(\frac{4}{n} + \frac{2}{n^2} + L_{33}\left(\frac{32}{3n^3}\right)\right)\log(y) + \\ \qquad + \frac{2\log(2) - 9\log(n)}{n} + \frac{2\log(2) - 11\log(n)}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right) \\ \frac{2Ru_2}{I} + \frac{Ru_3}{I} = \left(\frac{4}{n} + \frac{2}{n^2} + L_{33}\left(\frac{32}{3n^3}\right)\right)\log(y) + 15\log^2(n) + \\ \qquad + \frac{17\log(n) - \log(5)}{n} + \frac{27\log(n) - \log(5) + 8}{2n^2} + L_{33}\left(\frac{0.1}{n^2}\right) \end{cases} \tag{5.12}$$

As $y \geq 2$ and $\frac{v_j}{I}$ is an integer, we have $\frac{Rv_j}{I} \geq R$. □

Lemma 5.5. *For $n \geq 3309770$, the equation* (4.1) *has no solutions, except the trivial solutions.*

Proof. We consider the linear forms in logarithms defined by

(5.13)
$$\tilde{\Lambda}_j = \tilde{A}_j \log(\alpha) + \tilde{B}_j \log(\beta) + \log(\lambda_j) = \log|1+\tau_j|.$$

where $j = 1, 4$, with

(5.14)
$$\tilde{A}_1 = \frac{2u_3}{I},\ \tilde{B}_1 = -\frac{2u_2}{I},\ \lambda_1 = \frac{\alpha+\beta}{\alpha+1/\beta},\ \tau_1 = \frac{\gamma_1}{\gamma_2}\left(\frac{\beta^2-1}{\alpha\beta+1}\right),$$

and

(5.15)
$$\tilde{A}_4 = \frac{2u_2}{I},\ \tilde{B}_4 = \frac{2u_3}{I},\ \lambda_4 = \frac{1/\alpha+1/\beta}{\alpha+1/\beta},\ \tau_4 = -\frac{\gamma_4}{\gamma_3}\left(\frac{\alpha-1/\alpha}{\alpha+1/\beta}\right).$$

We obtain

(5.16)
$$\log\left|\tilde{\Lambda}_j\right| \leq -4\log(y) + \log(8) - a_j\log(n) - \frac{2}{n},$$

with $a_j = 4$ if $j = 1$ and $a_j = 3$ if $j = 4$.

$$\log\left|5\tilde{\Lambda}_j\right| \geq -6231.04\left(\log\left(b'\right)+.14\right)^2 h_1\, h_2; \tag{5.17}$$

with

(5.18)

$$\begin{cases} h_1 = \frac{9}{4}\log(n) + \frac{2}{n}, \\ h_2 = \left(\frac{3}{n\log(n)} + \frac{3}{2n^2\log(n)} - \frac{8}{5n^2\log(n)}\right)\log(y) \ + \ c_{j1}\log(n) + \frac{c_{j2}}{4n}, \\ b' = c_{j3} n\log(n), \end{cases}$$

with

$$c_{11} = \frac{15}{4},\ c_{12} = 49,\ c_{13} = 0.664,$$

$$c_{41} = 15,\ c_{42} = 37,\ c_{43} = 0.166$$

for $n \geq 50$. □

Second Step

We used a computational method based on Baker's method.

For $j = 1, 4$ and $50 \leq n \leq 3309770$, We developed program in Maple 9.5 and ran it on a Pentium 4 with 3.92 GHz running under Linux 7.2.

- First we consider $j = 1, 4$ (together) and $50 \leq n \leq 3045748$,

- then $j = 1$ and $3045748 \leq n \leq 3309770$. It took in average 1.75 seconds for each value of n.

Third Step

For $1 \leq n \leq 50$, we use Kash to obtain the solutions in Theorem 4.1. This completes the proof of this theorem.