Algorithmic and Theoretical Considerations for Computing Generators of the Centralizer of K in $\mathcal{U}(\mathfrak{g})$

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Number Theory Seminar
Thursday March 12, 2009

Outline

1. Outline
2. Preliminaries
3. Generators for $\mathfrak{U}(\mathfrak{g})^K$
4. Jackson-N Algorithms
5. Examples

Outline of the Talk

- Representation of a reductive Lie group

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- Universal Enveloping Algebra

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- Problem Statement and Context

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Theorem

G non-compact, semisimple. Then any non trivial irreducible π is of infinite dimension.

Unitary Dual

The set of irreducible unitary representations (**The Unitary Dual**) of G denoted by $\hat{G}$ is a fundamental tool to understand the actions of G. For G compact $\hat{G}$ is essentially determined. Barbasch has treated the classical complex groups. However the following cases are still not resolved:

Type A: $SU(p,q)$ for $(p,q > 2)$
Type B: $SO(p,q)$ for $(p,q \geq 3)$
Type C: $Sp(p,q)$ for $(p,q \geq 2)$
Type D: $SO(p,q)$ for $(p,q \geq 3)$, $SO^*(2n)$ for $(n \geq 4)$
Type F_4: $F_4(\mathbb{C})$, $F_4(split)$
Type E_6: $E_6(\mathbb{C})$, $E_6(split)$, $E_6(Hermitian)$, $E_6(quaternionic)$
Type E_7: $E_7(\mathbb{C})$ and all real non-compact forms
Type E_8: $E_8(\mathbb{C})$ and all real non-compact forms

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Here the fundamental object is the (g, K)-module, a vector space equipped with two compatible actions of $\mathfrak{g}$, the complexification of the Lie algebra of G and K. One is usually interested in Harish-Chandra modules that is (g, K)-modules that have finite multiplicities as representations of K.

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• **An attempt to determine irreducible (g, K)-modules (up to infinitesimal equivalence) by the action of $\mathfrak{U}(\mathfrak{g})^K$, the centralizer of the complexified K in the enveloping algebra of $\mathfrak{g}$, on any K-primary component.** Harish-Chandra, [Lepowski, McCollum 1973].

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Complete Results for: $SU(2,2), SU(n,1), SO(n,1)$. (Very few cases indeed)

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$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: a complexified Cartan decomposition of a complex semisimple Lie algebra $\mathfrak{g}$.
K: the subgroup of the adjoint group of $\mathfrak{g}$ corresponding to $\mathfrak{k}$.
Define a filtration

$$\mathfrak{U}(\mathfrak{g}) = \bigcup_{i=0}^{\infty} (\mathfrak{U}(\mathfrak{g}))_i$$

where $(\mathfrak{U}(\mathfrak{g}))_i$ is the span of all j-fold products of elements of $\mathfrak{g}$ for $j \leq i$.

By the Poincaré-Birkhoff-Witt theorem, the associated graded algebra with respect to this filtration is the symmetric algebra $S(\mathfrak{g})$. This is canonically isomorphic to the algebra of polynomial functions $\mathbb{C}[\mathfrak{g}^*]$, and since $\mathfrak{g}$ is reductive it is self-dual and we can identify $S(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{g}]$. In particular, if we can find generators for $\mathbb{C}[\mathfrak{g}]^K$ then any set of liftings of these generators to $\mathfrak{U}(\mathfrak{g})$ will generate $\mathfrak{U}(\mathfrak{g})^K$.

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$$S(\mathfrak{g})^K = S(\mathfrak{g})^K_r \text{ where } r = \binom{2\dim \mathfrak{g}}{2} \dim \mathfrak{p}$$

$S(\mathfrak{g})^K_r$ the subalgebra of $S(\mathfrak{g})^K$ defined by K-invariant polynomials of degree at most r.

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An analysis shows that for $SL_3(\mathbb{R})$ matrices with 10^{32} entries would be needed in order to implement Kostant's method.

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A Molien series for $\mathfrak{U}(\mathfrak{g})^K$

We describe a method by which Kostant's algorithm can be significantly accelerated by exploiting the Kostant-Rallis theorem via a certain homomorphism from $\mathfrak{U}(\mathfrak{g})^K$ to the ring of regular functions on the nilpotent cone in $\mathfrak{p}$. The situation is analogous to that in the invariant theory of finite groups, where the Molien series is used to accelerate the algorithm suggested by Noether's degree bound.

Harm Derksen and Gregor Kemper:*Computational invariant theory, Invariant Theory and Algebraic Transformation Groups, I*, Springer-Verlag, Berlin, 2002, Encyclopaedia of Mathematical Sciences, 130.

A Molien series for $\mathfrak{U}(\mathfrak{g})^K$

Various structural and algebraic considerations yield:

$$\mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{k}] \otimes \mathbb{C}[\mathfrak{p}] \simeq \mathbb{C}[\mathcal{N}_\mathfrak{k}] \otimes \mathbb{C}[\mathfrak{k}]^K \otimes \mathbb{C}[\mathcal{N}_\mathfrak{p}] \otimes \mathbb{C}[\mathfrak{p}]^K$$

where $\mathcal{N}_\mathfrak{k}$ and $\mathcal{N}_\mathfrak{p}$ are the cone of nilpotent elements of $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Hence

$$\mathbb{C}[\mathfrak{g}]^K \simeq (\mathbb{C}[\mathcal{N}_\mathfrak{k}] \otimes \mathbb{C}[\mathcal{N}_\mathfrak{p}])^K \otimes \mathbb{C}[\mathfrak{k}]^K \otimes \mathbb{C}[\mathfrak{p}]^K.$$

This is not an algebra isomorphism but we know that lifting of generators will work!!

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Theorem (Jackson, N)

$\mathcal{A} \simeq \mathbb{C}[\mathcal{N}_{\mathfrak{p}}]^{K^e}$, where e is a regular nilpotent element of $\mathfrak{k}$ and K^e, the isotropy group of e in K.

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This is a grading transferable to $\mathcal{A}$ via the above isomorphism. Define a *Molien series* for $\mathfrak{U}(\mathfrak{g})^K$ as follows:

$$M(t) = \sum_{i=0}^{\infty} (\dim \mathcal{A}_i) t^i.$$

We will see that $\dim \mathcal{A}_i$ is finite.

Computing $M(t)$

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Computing the Molien series can be reduced to computation of M-invariants on $\mathfrak{k}$ together with a Gröbner basis calculation. Since M is frequently the product of a finite group and a torus, this often reduces the computation to familiar algorithms from the invariant theory of finite groups and integer programming.

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Theorem (Jakson, N)

The formal power series $N(t) = \sum_{i=0}^{\infty} (\dim \mathbb{C}[\mathcal{N}_{\mathfrak{k}}]_i^M)\, t^{2i}$ *coincides with the Molien series* $M(t)$. *In particular, the coefficients of* $M(t)$ *are finite.*

Computing $M(t)$

Let k be the rank of $\mathfrak{k}$ and $u_1, \ldots, u_k$ a system of homogeneous generators for $\mathbb{C}[\mathfrak{k}]^K$ of degrees $d_1, \ldots, d_k$ respectively. Then $(u_1, \ldots, u_k)$ is a regular sequence in $\mathbb{C}[\mathfrak{k}]$. In other words codim $\mathcal{N}_{\mathfrak{k}}$ in $\mathfrak{k}$ is k.

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Theorem (Jackson, N)

$$M(t) = P(t^2) \prod_{i=1}^{k} (1 - t^{2d_i}).$$

Computing $M(t)$

M, the cenralizer of $\mathfrak{a}$ in K, is reductive; hence it has a compact real form $M_{\mathbb{R}}$. When M is abelian we can define $R(x_1, \ldots, x_n)$ by the formula

$$R(x_1, \ldots, x_n) = \int_{m \in M_{\mathbb{R}}} \prod_{i=1}^{n} \frac{1}{1 - mx_i}\, dm.$$

Then $P(t) = R(t, \ldots, t)$.

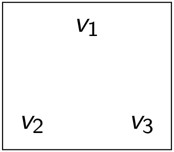
Computing $M(t)$

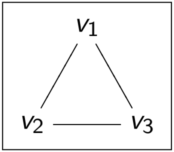
If the pair (G, K) corresponds to a split real group, then M is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^l$, where l is the rank of G. In particular, M is abelian, and the invariant integral described in the previous section collapses to a finite sum. Since M is a 2-group, we see that x_i^2 is M-invariant for all i. Now let $\mathcal{S}$ denote the set of all M-invariant square-free monomials in $x_1, \ldots, x_n$. Then

$$R(x_1, \ldots, x_n) = \frac{\sum_{\mu \in \mathcal{S}} \mu}{\prod_{i=1}^{n} (1 - x_i^2)}.$$

Computing $M(t)$ for $SL_{\mathbb{R}}(3)$

The square-free invariant monomials correspond to the graphs:





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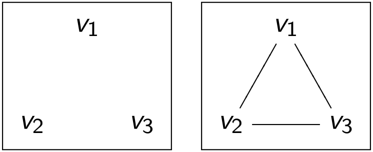

$$P(t) = \frac{1+t^3}{(1-t^2)^3}.$$

Since $k = 1$ and $d_1 = 2$,

$$M(t) = \frac{1+t^6}{(1-t^4)^2}.$$

Computing $M(t)$ for $SL_{\mathbb{R}}(4)$

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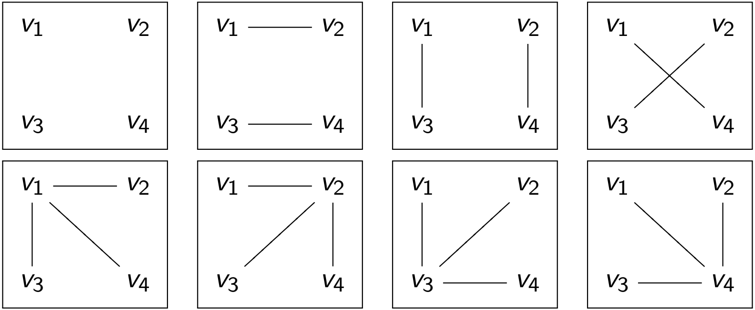

Computing $M(t)$ for $SL_{\mathbb{R}}(4)$

together with their edge-complements (i.e. the graphs obtained from those above by deleting all existing edges and placing an edge between each pair of vertices which were previously unconnected).

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$$P(t) = \frac{1 + 3t^2 + 8t^3 + 3t^4 + t^6}{(1 - t^2)^6}.$$

Here $k = 2, d_1 = 2$, and $d_2 = 2$, so we obtain

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This technique is valid for $SL_{\mathbb{R}}(n)$.

Algorithms

If one knows the Molien series $M(t)$ then one has an algorithm to test whether a given collection of K-invariants $f_1, \ldots, f_n \in \mathbb{C}[\mathfrak{g}]$ (together with the usual generators for $\mathbb{C}[\mathfrak{k}]^K$ and $\mathbb{C}[\mathfrak{p}]^K$) generate $\mathbb{C}[\mathfrak{g}]^K$. Let $\pi : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathcal{N}(\mathfrak{k})] \otimes \mathbb{C}[\mathcal{N}(\mathfrak{p})]$ be the projection, and let $\mathcal{A}'$ be the subalgebra of $\mathcal{A}$ generated by $\pi(f_1), \ldots, \pi(f_n)$.

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Replacing the f_i by their $\mathfrak{k}$-homogeneous components if necessary, define a formal power series $Q(t)$ by

$$Q(t) = \sum_{i=0}^{\infty} (\dim A'_i)\, t^i.$$

If $Q(t) = M(t)$, then $f_1, \ldots, f_n$ is included in a set of generators for $\mathfrak{U}(\mathfrak{g})^K$.

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- Computing $M(t)$ (or, equivalently, computing $P(t)$)
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We have two ways of manufacturing elements of $\mathbb{C}[\mathfrak{g}]^K$.

Method I: Linear Algebra

Method I is general and produces a basis for $\mathbb{C}[\mathfrak{g}]_d^K$ by computing the Kernel of a matrix. It is suitable for implementation on a computer algebra system. If $M(t)$ is known, this leads immediately to an algorithm which computes generators for $\mathbb{C}[\mathfrak{g}]^K$: starting with $i = 0$, we increment i until a basis for $\sum_{d=0}^{i} \mathbb{C}[\mathfrak{g}]_d^K$ gives $Q(t) = M(t)$.

Method II: Trace Forms

Method II is much faster than Method I. But there is no guarantee that $\mathbb{C}[\mathfrak{g}]^K$ will be generated by trace forms.

Method II: Trace Forms

Method II is much faster than Method I. But there is no guarantee that $\mathbb{C}[\mathfrak{g}]^K$ will be generated by trace forms.
In general, one can always decompose $\mathfrak{g}$ as a sum of irreducible K-modules:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$$

for some m. For $1 \leq i \leq m$ let π_i denote the K-equivariant projection from $\mathfrak{g}$ to $\mathfrak{g}_i$. Passing to a representation V of $\mathfrak{g}$, we can regard each $\mathfrak{g}_i$ as a space of matrices on which K acts by conjugation. Now for any sequence $i_1, \ldots, i_d$ with $1 \leq i_j \leq m$, define a function $f_{V,i_1,\ldots,i_d} : \mathfrak{g} \to \mathbb{C}$ by the formula

$$f_{V,i_1,\ldots,i_d}(x) = trace_V(\pi_{i_1}(x) \ldots \pi_{i_d}(x)).$$

Evidently $f_{V,i_1,\ldots,i_d}$ is a polynomial of degree d, and by construction it lies in $\mathbb{C}[\mathfrak{g}]^K$.

Outline

1. Outline
2. Preliminaries
3. Generators for $\mathfrak{U}(\mathfrak{g})^K$
4. Jackson-N Algorithms
5. Examples

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generate $\mathbb{C}[\mathfrak{g}]^K$.
In other words, letting A and S be the antisymmetric and symmetric parts, respectively, of a generic element $x \in \mathfrak{g}$, liftings of the polynomial functions

$$\{tr(A^2), tr(S^2), tr(A^2S), tr(S^3), tr((AS)^2), tr(ASA^2S^2)\}$$

generate $\mathfrak{U}(\mathfrak{g})^K$.

$SL_{\mathbb{R}}(4)$

Letting V denote the standard representation of $\mathfrak{sl}_4$, we can check that $\mathfrak{U}(\mathfrak{g})^K$ is generated by liftings of trace forms on V of degree nine or less.

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Then define

$$\pi_1(x) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \pi_2(x) = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \quad \pi_3(x) = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \quad \pi_4(x) = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$$

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generate $\mathbb{C}[\mathfrak{g}]^K$,
whence liftings of the polynomial functions

$$\{tr(A), tr(D), tr(A^2), tr(D^2), tr(BC), tr(ABC), tr(BDC), tr(ABDC), tr((BC)^2), tr(ABDCBC)\}.$$

generate $\mathfrak{U}(\mathfrak{g})^K$. (This agrees with Zhu's result.)

$G_{2(2)}$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ be the short and long simple roots, respectively. Let X_α be the element of a Chevalley basis corresponding to the root α, and put $H_\alpha = [X_\alpha, X_{-\alpha}]$. Conjugating by an inner automorphism if necessary, we may take

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$$\mathfrak{k} = span(X_{\alpha_1}, X_{-\alpha_1}, H_{\alpha_1}, X_{3\alpha_1+2\alpha_2}, X_{-3\alpha_1-2\alpha_2}, H_{3\alpha_1+2\alpha_2})$$
$$\mathfrak{p} = span(X_{\alpha_2}, X_{-\alpha_2}, X_{\alpha_1+\alpha_2}, X_{-\alpha_1-\alpha_2}, X_{2\alpha_1+\alpha_2}, X_{-2\alpha_1-\alpha_2}, X_{3\alpha_1+\alpha_2}, X_{-3\alpha_1-\alpha_2}).$$

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$$M(t) = \frac{1 + 3t^4 + 8t^6 + 3t^8 + t^{12}}{(1 - t^4)^4}$$

$G_{2(2)}$

Put

$$\mathfrak{g}_1 = span(X_{\alpha_1}, X_{-\alpha_1}, H_{\alpha_1})$$
$$\mathfrak{g}_2 = span(X_{3\alpha_1+2\alpha_1}, X_{-3\alpha_1-2\alpha_2}, H_{3\alpha_1+2\alpha_2})$$
$$\mathfrak{g}_3 = \mathfrak{p}$$

so that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ is the decomposition of $\mathfrak{g}$ into irreducible K-modules.

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Let V be the seven-dimensional irreducible representation of $\mathfrak{g}$ induced from the embedding of $\mathfrak{g}_2$ into $\mathfrak{so}_7$.

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One finds that the trace forms

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generate $\mathbb{C}[\mathfrak{g}]^K$.

Recent work of Steven Jackson

Recently, Steven Jackson (working alone) seems to have developed a theory for writing down the generators of $\mathfrak{U}(\mathfrak{g})^K$ for any split real form $\mathfrak{g}_{\mathbb{R}}$ using root systems.

No computer calculations is required in the process.