Regular Maps and a Prelude to Morphisms of Varieties

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Up until this point we have studied a wide variety (pun intended) of varieties, but as there are many different types (affine, projective, quasi-affine, etc...) and come with many different properties, is there a way for them to interact? What sorts of maps are allowed between them? Ultimately we wish to define Morphisms between varieties in order to establish a category with which to work.

Before we can actually develop a morphism, we introduce the notation of a Regular Map. These are maps from a variety into the ground field k. As these are defined by polynomial and rational expressions, we must give two alternate definitions for when the variety in question is affine or projective to be careful of the choice of homogeneous polynomials in the latter case.

Definition. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety. A map $f : Y \to k$ is said to be Regular at a point $P \in Y$ if there exists an open neighborhood, U, of P wholly contained in Y so that there exist polynomials $g, h \in k[x_1, \ldots, x_n]$ with h nowhere vanishing on U so that $f = \frac{g}{h}$ for all points in U. Additionally, we say that a map f is regular on Y if it is regular for all points of Y.*

Definition. *Let $Y \subseteq \mathbb{P}^n$ be a projective variety. A map $f : Y \to k$ is said to be Regular at a point $P \in Y$ if there exists an open neighborhood, U, of P wholly contained in Y so that there exist homogeneous polynomials $g, h \in k[x_0, \ldots, x_n]$ of the same degree, with h nowhere vanishing on U so that $f = \frac{g}{h}$ for all points in U. Additionally, we say that a map f is regular on Y if it is regular for all points of Y.*

We note the necessity that g and h are of the same degree in the latter definition in order to be a well defined function on $\mathbb{P}^n$. Then since g and h are of the same degree, say d, then

$$\frac{f(\lambda a_0, \ldots, \lambda a_n)}{g(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d f(a_0, \ldots, a_n)}{\lambda^d g(a_0, \ldots, a_n)} = \frac{f(a_0, \ldots, a_n)}{g(a_0, \ldots, a_n)}$$

For any nonzero $\lambda \in k$, so this does indeed provide a well-defined function on $\mathbb{P}^n$.

Examples.

(1) If we consider all of $\mathbb{A}^n$ as a variety over itself, the only regular functions on $\mathbb{A}^n$ are polynomials in the n variables. Additionally the only regular functions on $\mathbb{P}^n$ are constants.

(2) If $Y = Z(x^2+y^2-1)$ is our affine variety, then $f(x,y) = \frac{1-y}{x}$ is regular at $(0,1)$. Since

$$\frac{1-y}{x} = \frac{1-y}{x} \cdot \frac{1+y}{1+y} = \frac{1-y^2}{x(1+y)} = \frac{x^2}{x(1+y)} = \frac{x}{1+y}$$

Proposition 1. *If Y is an affine, quasi-affine, or projective variety, a regular map $f : Y \to k$ is continuous when k is identified with $\mathbb{A}^1$ under the Zariski Topology.*

We shall only show the case when Y is an affine variety and leave the proof of the result for a projective variety as an exercise to the reader (as it is very similar).

Lemma 2. *A subset Z of a topological space is closed if and only if X can be covered by open sets U such that $Z \cap U$ is closed in U for every such U.*

Now let us proceed with the proof of the proposition.

Proof. There are many equivalent definitions for continuous, but we will show that f^{-1} is a closed map. Since a (nontrivial) closed set of $\mathbb{A}^1$ is just a finite set of points (Example 1.1.1), it will suffice to show that for any point $a \in \mathbb{A}^1$, $f^{-1}(a)$ is closed in Y. We shall use the lemma above. Let U be such an open set so that $f = \frac{g}{h}$ for $g, h \in k[x_1, \ldots, x_n]$ and h non-vanishing on U. Then

$$\begin{aligned} f^{-1}(a) \cap U &= \left\{ P \in Y \;\middle|\; \frac{g(P)}{h(P)} = a \right\} \\ &= \{P \in Y \mid g(P) - ah(P) = 0\} \\ &= \{P \in Y \mid P \in Z(g - ah)\} \\ &= U \cap Z(g - ah) \end{aligned}$$

which is of course closed in U under the subspace topology. Thus f^{-1} is indeed a closed map so f is continuous.

□

Remark 3. *Let f and g be regular functions on a variety Y. Then if there exists a nonempty open subset of Y such that $f = g$ on U, then $f = g$ everywhere in Y. This follows from the fact that $Z(f-g)$ is closed in Y, but also $U \subseteq Z(f-g)$. Then since the closure of U is the smallest closed set containing U, it follows that $Y = \overline{U} \subseteq Z(f-g)$ as U is dense in Y (as Y is a variety and hence irreducible). Thus $Z(f-g) = Y$.*

Let us clear up the abusive of notation in the previous remark. Even though f and g are not polynomials, they behave as rational functions on two open sets and hence on their nonempty intersection. Then we may notate $Z(f-g)$ to be the zero set of the numerator of the difference of rational functions fter getting common denominators. Now that we know a few facts about regular maps, let us try to created something greater out of them.

Definition. *Let Y be any variety. We define <u>Ring of Regular Functions</u> on Y, $\underline{\mathcal{O}(Y)}$ to be the set of all regular functions on Y. Notice that this does indeed have a ring structure via the usual addition and multiplication.*

Examples. *Following the previous examples*

(1) $\mathcal{O}(\mathbb{A}^n) = k[x_1, \ldots, x_n]$

(2) $\mathcal{O}(\mathbb{P}^n) = k$

Definition. *Additionally we define the* Local Ring of P, $\mathcal{O}_P$, *to be the ring of germs of regular functions of regular functions on Y near P. Hence an element of $\mathcal{O}_P$ is a pair $\langle U, f\rangle$ where U is an open neighborhood of P and f a regular function at P and we impose an equivalence relation that $\langle U, f\rangle \sim \langle V, g\rangle$ if $f = g$ on $U \cap V$.*

Before we continue, let us verify that this does indeed define an equivalence relation.

- Naturally $\langle U, f\rangle \sim \langle U, f\rangle$
- If $\langle U, f\rangle \sim \langle V, g\rangle$, then $f = g$ on $U \cap V$. Then $g = f$ on $V \cap U$ so $\langle V, g\rangle \sim \langle U, f\rangle$
- If $\langle U, f\rangle \sim \langle V, g\rangle$ and $\langle V, g\rangle \sim \langle W, h\rangle$, then $f = g$ on $U \cap V$ and $g = h$ on $V \cap W$. Then using the remark, $f = g$ on $U \cup V$ and $g = h$ on $V \cup W$. In particular $f = g$ on V and $g = h$ on V, so $f = h$ on all of U, hence $f = h$ on $U \cap W$. Thus $\langle U, f\rangle \sim \langle W, h\rangle$.

Next since Y is a variety and hence irreducible, any two nonempty open sets will have a nonempty intersection. Then we may impose a ring structure by letting addition and multiplication be the usual operations on the intersections of open subsets.

Notice that this is in fact a local ring. We claim that its unique maximal ideal is the set of germs of regular functions that vanish at P. Indeed if $f(P) \neq 0$, then one can find a neighborhood of P, V so that $\frac{1}{f}$ is regular and $\langle V, \frac{1}{f}\rangle$ is a multiplicative inverse. Thus this set is equal to $\mathcal{O}_P \backslash \mathcal{O}_P^*$. Then we are finished once we show that this set forms an ideal (Ulrich, proposition 1.7), but this is easy to see as this set contains 0, the sum of two vanishing functions vanishes, and a multiple of such a function vanishes as well.

Additionally, if we let $\mathfrak{m}$ denote this maximal ideal, $\mathcal{O}_P/\mathfrak{m} \cong k$. This can be seen easily by the map $\varphi : \mathcal{O}_P \to k$ where $\varphi : \langle U, f\rangle \mapsto f(P)$. This map is surjective as one can always find a polynomial g to assume a value in k. Then take g itself (as it is equal to $\frac{g}{1}$) to be the regular function and take the entire variety to be the open set. The kernel of this map is the set of germs which vanish at P, which is exactly $\mathfrak{m}$. Then by the first isomorphism $\mathcal{O}_P/\mathfrak{m} \cong k$.

Definition. *For any variety Y we define the* Function Field $K(Y)$ *to be the ring of all germs of regular functions. In other words, an element of $K(Y)$ is a pair $\langle U, f\rangle$ where U is an open subset of Y and f is regular on U and we identify two pairs $\langle U, f\rangle = \langle V, g\rangle$ if $f = g$ on $U \cap V$.*

Notice that $K(Y)$ is in fact a field. It is certainly a ring with a similar structure to that of $\mathcal{O}_P$. To show it is a field if $\langle U, f\rangle \in K(Y)$ with $f$nonzero, then $\frac{1}{f}$ is regular on $V = U \backslash U \cap Z(f)$. Hence $\langle V, \frac{1}{f}\rangle$ is a multiplicative inverse for this pair.

Notice that by proper restrictions, one can define natural injective maps $\mathcal{O}(Y) \hookrightarrow \mathcal{O}_P \hookrightarrow K(Y)$. For this reason we may regard $\mathcal{O}(Y)$ and $\mathcal{O}_P$ as subrings of $K(Y)$. We will soon see that, once we define what a morphism is, these objects will serve as invariants for varieties. Our next task is to introduce morphisms and relate these invariants to the coordinate ring of the variety in question.