ABSTRACT NONSINGULAR CURVES

Affine Varieties

Notation. Let k be a field, such as the rational numbers $\mathbb{Q}$ or the complex numbers $\mathbb{C}$. We call *affine n-space* the collection $\mathbb{A}^n(k)$ of points $P = \bigl(a_1, a_2, \ldots, a_n\bigr)$ with coordinates $a_i \in k$. For example, $\mathbb{A}^1(k) = k$. Similarly, we call *projective n-space* the collection $\mathbb{P}^n(k)$ of equivalence classes $\bigl(a_1 : \cdots : a_n : a_0\bigr)$, where we say two nonzero points $\bigl(a_1, \ldots, a_n, a_0\bigr)$, $\bigl(b_1, \ldots, b_n, b_0\bigr) \in \mathbb{A}^{n+1}(k)$ are equivalent if $a_i = \lambda\, b_i$ for some nonzero $\lambda \in k$. For example, $\bigl(a_1 : a_0\bigr) = \bigl(b_1 : b_0\bigr) \in \mathbb{P}^1(k)$ formally if and only if $\dfrac{a_1}{a_0} = \dfrac{b_1}{b_0}$. We can always embed $\mathbb{A}^n(k) \hookrightarrow \mathbb{P}^n(k)$ via the map $\bigl(a_1, \ldots, a_n\bigr) \mapsto \bigl(a_1 : \cdots : a_n : 1\bigr)$. We will be interested in subsets of these spaces.

Denote $k[x_1, x_2, \ldots, x_n]$ as the collection of polynomials in n variables with coefficients in k. Given a collection of polynomials $\{f_1, f_2, \ldots, f_m\}$, we denote the ideal

$$\langle f_1, f_2, \ldots, f_m \rangle = \left\{ g_1\, f_1 + g_2\, f_2 + \cdots + g_m\, f_m \;\middle|\; g_1, g_2, \ldots, g_m \in k[x_1, x_2, \ldots, x_n] \right\}.$$

Recall that $\wp = \langle f_1, f_2, \ldots, f_m \rangle$ is a prime ideal if and only if the quotient ring $\mathcal{O} = k[x_1, x_2, \ldots, x_n]/\wp$ is an integral domain. Whenever this is the case, we call an *affine variety* a set in the form

$$X = \left\{ P \in \mathbb{A}^n(k) \;\middle|\; f_1(P) = f_2(P) = \cdots = f_m(P) = 0 \right\}.$$

I claim that any $f \in \mathcal{O}$ is actually a function $f : X \mapsto \mathbb{P}^1(k)$. To see why, say that $f = h_1 + \wp = h_2 + \wp$ for some polynomials h_1, h_2. Since $h_1 - h_2 \in \wp$, we have $h_1 - h_2 = \sum_\alpha g_\alpha\, f_\alpha$, so that $h_1(P) - h_2(P) = \sum_\alpha g_\alpha(P)\, f_\alpha(P) = 0$. Hence $f(P) = \bigl(h_1(P) : 1\bigr) = \bigl(h_2(P) : 1\bigr)$ is a well-defined element. We are motivated by the following philosophy, usually attributed to Alexander Grothendieck:

Instead of studying a variety X, study its ring of functions $\mathcal{O}$.

Zariski Topology. We wish to place a topology on an affine variety X. We say that a subset $V \subseteq X$ is *closed* if there exists a subset $T \subseteq k[x_1, x_2, \ldots, x_n]$ such that

$$V = Z(T) = \left\{ P \in X \;\middle|\; f(P) = 0 \text{ for all } f \in T \right\}.$$

T may be taken to be an ideal containing $\wp$, but this is not crucial. We say that a subset $U \subseteq X$ is *open* if $U = X - V$ is the compliment of some closed set V. I claim that:

- Both $\emptyset$ and X are open sets.
- If $\{U_\alpha\}$ is a arbitrary collection of open sets that $\bigcup_\alpha U_\alpha$ is also open.
- If $\{U_\alpha\}$ is a finite collection of open sets, then $\bigcap_\alpha U_\alpha$ is also open.

Let me explain why. It is easy to see that $\emptyset = X - Z(\{0\})$ and $X = X - Z(\{1\})$. Say that each $U_\alpha = X - V_\alpha$ for some closed set $V_\alpha = Z(T_\alpha)$. De Morgan's Laws imply that

$$\bigcup_\alpha U_\alpha = X - \bigcap_\alpha V_\alpha = X - Z\left(\bigcup_\alpha T_\alpha\right),$$

$$\bigcap_\alpha U_\alpha = X - \bigcup_\alpha V_\alpha = X - Z\left(\prod_\alpha T_\alpha\right).$$

This topology is called the *Zariski topology*, named after Oscar Zariski.

Relation with Complex Analysis. Here are how the definitions we've studied before match up with the ones gives thus far:

Complex Analysis	Commutative Algebra
$\mathbb{C}$, the ambient space	$\mathbb{A}^1(k)$, affine space
D, region in $\mathbb{C}^n$	X, affine variety in $\mathbb{P}^n(k)$
Ω, open set	$U = X - Z(T)$, Zariski open set
$f : D \to \mathbb{C}$, analytic function	$f : X \to \mathbb{P}^1(k)$, rational function
$\mathfrak{S}$, sheaf of analytic functions	$\mathcal{O}$, ring of rational functions

Example. We seek a working definition for the types of maps allowed between varieties. To this end, we consider a simple example.

Let k be a field such that $\sqrt{-1} \notin k$. Let $f_1(x, y) = x^2 + y^2 - 1$ be a polynomial from the ring $k[x, y]$, and consider the following affine variety:

$$X = \left\{ (x, y) \in \mathbb{A}^2(k) \,\middle|\, x^2 + y^2 - 1 = 0 \right\}.$$

This is simply the unit circle. There is a morphism $\psi : \mathbb{P}^1(k) \to X$ defined by

$$(a_1 : a_0) \mapsto \left(\frac{a_1^2 - a_0^2}{a_1^2 + a_0^2}, \ \frac{2\, a_1\, a_0}{a_1^2 + a_0^2} \right).$$

Note that this map is well-defined and only involves ratios of polynomials. We will show that this map is *birational* by exhibiting another morphism $\varphi : X \to \mathbb{P}^1(k)$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$ a maps also involving ratios of polynomials.

To this end, consider the following subsets of X:

$$U_1 = \left\{ (x, y) \in X \,\middle|\, (x, y) \neq (-1, 0) \right\} = X - Z(\{x + 1\}),$$

$$U_2 = \left\{ (x, y) \in X \,\middle|\, (x, y) \neq (+1, 0) \right\} = X - Z(\{x - 1\}).$$

These are open sets such that $X = U_1 \cup U_2$, so they define an open cover of the unit circle. Define now the following maps:

$$\left.\begin{array}{ll} \varphi_1 : & U_1 \to \mathbb{P}^1(k) \\ \varphi_2 : & U_2 \to \mathbb{P}^1(k) \end{array}\right\} \quad \text{which send} \quad \left\{\begin{array}{l} (x, y) \mapsto (1 + x : y) \\ (x, y) \mapsto (y : 1 - x) \end{array}\right.$$

Using the formal relation

$$x^2 + y^2 = 1 \qquad \Longleftrightarrow \qquad \frac{1+x}{y} = \frac{a_1}{a_0} = \frac{y}{1-x} \qquad \Longleftrightarrow \qquad (x,y) = \left(\frac{a_1^2 - a_0^2}{a_1^2 + a_0^2},\ \frac{2\,a_1\,a_0}{a_1^2 + a_0^2} \right)$$

we immediately see that $\varphi_1\big|_{U_1 \cap U_2} = \varphi_2\big|_{U_1 \cap U_2}$. In particular, the pairs $\langle U_1,\, \varphi_1 \rangle \simeq \langle U_2,\, \varphi_2 \rangle$ are equivalent, so they define a morphism

$$\varphi : X \to \mathbb{P}^1(k) \qquad \text{which sends} \qquad (x,y) \mapsto \begin{cases} \big(1 + x : y\big) & \text{if } (x,y) \in U_1, \text{ and} \\ \big(y : 1 - x\big) & \text{if } (x,y) \in U_2. \end{cases}$$

It is easy to verify that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$. Note how we *glued* two maps together to find a morphism.

Abstract Nonsingular Varieties

Points as Primes. Recall that Grothendieck's philosophy is to replace X with $\mathcal{O}$ whenever possible. But what exactly is the relationship between them?

Assume now that k is algebraically closed. Fix a prime ideal $\wp = \langle f_1,\, f_2,\, \dots,\, f_m \rangle$ in $k[x_1,\, x_2,\, \dots,\, x_n]$; then $\mathcal{O} = k[x_1,\, x_2,\, \dots,\, x_n]/\wp$ is an integral domain. Continue to denote X as the collection of $P \in \mathbb{A}^n(k)$ such that $f(P) = 0$ for all $f \in \wp$.

Proposition 1. *There is a one-to-one correspondence between points $P = \big(a_1,\, a_2,\, \dots,\, a_n\big)$ on X and maximal ideals $\mathfrak{m}_P = \langle x_1 - a_1,\, x_2 - a_2,\, \dots,\, x_n - a_n \rangle$ in mSpec $\mathcal{O}$.*

Proof. Fix a point $P = \big(a_1,\, a_2,\, \dots,\, a_n\big)$ on X. We show that $\mathfrak{m} = \langle x_1 - a_1,\, x_2 - a_2,\, \dots,\, x_n - a_n \rangle$ is a maximal ideal. Recall that an ideal $\mathfrak{m} \subseteq \mathcal{O}$ is maximal if and only if $\mathcal{O}/\mathfrak{m}$ is a field. Since $\mathcal{O}/\mathfrak{m} \simeq k[a_1, \dots, a_n] = k$, we see that $\mathfrak{m}$ is indeed maximal.

Conversely, fix a maximal ideal $\mathfrak{m}$ in mSpec $\mathcal{O}$. We show that $\mathfrak{m}_P = \langle x_1 - a_1,\, x_2 - a_2,\, \dots,\, x_n - a_n \rangle$ for some point $P = \big(a_1,\, a_2,\, \dots,\, a_n\big)$ on X. Since $\mathcal{O}/\mathfrak{m}$ is a finite extension of k and k is algebraically closed, we have a map

$$\mathcal{O} \longrightarrow \mathcal{O}/\mathfrak{m} \xrightarrow{\ \sim\ } k.$$

Let $a_i \in k$ be the image of $x_i \in \mathcal{O}$, and consider the ldeal $\mathfrak{m}_P = \langle x_1 - a_1,\, x_2 - a_2,\, \dots,\, x_n - a_n \rangle$. Since each polynomial $x_i - a_i$ is in the kernel of this map, we must have $\mathfrak{m}_P \subseteq \mathfrak{m} \subseteq \mathcal{O}$. But $\mathfrak{m}_P$ is a maximal ideal, so $\mathfrak{m} = \mathfrak{m}_P$. □

Abstract Varieties. To recap, there is an injective map $X \simeq \mathrm{mSpec}\,\mathcal{O} \hookrightarrow \mathrm{Spec}\,\mathcal{O}$, for some integral domain $\mathcal{O}$, which is defined by $P \mapsto \mathfrak{m}_P$. Even though is map not surjective, for any integral domain $\mathcal{O}$ we say that $\mathrm{Spec}\,\mathcal{O}$ is a *abstract variety.* Note that $X \simeq \mathrm{mSpec}\,\mathcal{O}$ is really an affine variety, although we will abuse notation and call $\mathrm{Spec}\,\mathcal{O}$ an affine variety as well.

Example. Let $\mathcal{O} = \mathbb{Z}$. The only prime ideals in $\mathcal{O}$ are $\mathfrak{m} = p\,\mathbb{Z}$ and $\{0\}$ for any rational prime $p \in \mathbb{Z}$, because

$$\mathcal{O}/\mathfrak{m} \simeq \begin{cases} \mathbb{F}_p & \text{for } \mathfrak{m} = p\,\mathbb{Z}, \text{ and} \\ \mathbb{Z} & \text{for } \mathfrak{m} = \{0\}. \end{cases}$$

In particular, $\mathrm{mSpec}\,\mathcal{O} \simeq \{2,\, 3,\, 5,\, \dots,\, p,\, \dots\}$ consists of the rational primes and $\mathrm{Spec}\,\mathcal{O} = \{0\} \cup \mathrm{mSpec}\,\mathcal{O}$. Hence the affine variety $\mathrm{mSpec}\,\mathcal{O}$ has points which are the rational primes, yet the abstract variety Spec $\mathcal{O}$ is strictly larger. These are varieties – but we cannot express the points as zeroes of some polynomial!

Zariski's Notion of Nonginsgularity. Let's continue to denote X as the collection of $P \in \mathbb{A}^n(k)$ such that $f(P) = 0$ for all $f \in \wp$. We say that X is a *nonsingular affine variety* if we can define a tangent at every point $P \in X$. We can make this rigorous using derivatives as follows: Since X is the collection of zeroes for the set $\wp = \langle f_1, f_2, \ldots, f_m \rangle$, we define the Jacobian of X at a point P to be the $m \times n$ matrix

$$J\big(f_1, f_2, \ldots, f_m\big)(P) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(P) & \dfrac{\partial f_1}{\partial x_2}(P) & \cdots & \dfrac{\partial f_1}{\partial x_n}(P) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(P) & \dfrac{\partial f_m}{\partial x_2}(P) & \cdots & \dfrac{\partial f_m}{\partial x_n}(P) \end{bmatrix}.$$

Since $\wp$ is a prime ideal, the number of polynomials cannot be greater than the number of variables: $m \leq n$. The rank of this matrix should be as large as possible, so we say that X is a nonsingular affine variety of $\operatorname{rank}_k J\big(f_1, f_2, \ldots, f_m\big)(P) = m$ for every $P \in X$.

Points as DVRs. Keeping Grothendieck's philosophy in mind, we wish to translate this idea into one involving the ring $\mathcal{O}$. We give some motivation: Say that X is the collection of $P \in \mathbb{A}^n(k)$ such that $f(P) = 0$ for all $f \in \wp$. If $P = \big(a_1, a_2, \ldots, a_n\big)$ is a point on X, then any f in $\mathcal{O} = k[x_1, x_2, \ldots, x_n]/\wp$ has a Taylor series expansion

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = f(P) &+ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,(x_1 - a_1) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2}\,\frac{\partial^2 f}{\partial x_i\,\partial x_j}\,(x_i - a_i)\,(x_j - a_j) + \cdots \end{aligned}$$

Modulo the maximal ideal $\mathfrak{m}_P = \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle$, we have $f(Q+P) \equiv f(P) \mod \mathfrak{m}_P$ and $f(Q+P) \equiv f(P) + \nabla f(P) \cdot Q \mod \mathfrak{m}_P^2$ for any $Q \in X$. This means

- $f \mod \mathfrak{m}_P$ is the same as computing $f(P)$.
- $f \mod \mathfrak{m}_P^2$ is the same as computing $\nabla f(P)$.

For now, assume that $n = 2$ and $m = 1$, i.e., $\mathcal{O} = k[x, y]/\big(f_1\big)$. The following proposition explains then importance of $\mathfrak{m}_P/\mathfrak{m}_P^2$ in associating $P \in X$ with a discrete valuation ring. It is a restatement of Proposition 9.2 on pages 94-95 in Atiyah-Macdonald.

Theorem 2. *For each $P \in X$, the following are equivalent:*

i. *X is nonsingular at P, i.e., $\nabla f_1(P) \neq (0,0)$.*
ii. $\dim_k\big(\mathfrak{m}_P/\mathfrak{m}_P^2\big) = 1$.
iii. *$\mathfrak{m}_P\,\mathcal{O}_P$ is a principal ideal.*
iv. *$\mathcal{O}_P$, the localization of $\mathcal{O}$ at $\mathfrak{m}_P$, is a discrete valuation ring.*
v. *$\mathcal{O}_P$ is integrally closed.*

Proof. (i) $\iff$ (ii). We make a general observation, following Zariski. Say that we have $\mathcal{O} = k[x_1, x_2, \ldots, x_n]/\wp$, as before. For each point $P \in X$, we define the Jacobian to be the

$m \times n$ matrix

$$J\big(f_1,\, f_2,\, \ldots,\, f_m\big)(P) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(P) & \dfrac{\partial f_1}{\partial x_2}(P) & \cdots & \dfrac{\partial f_1}{\partial x_n}(P) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(P) & \dfrac{\partial f_m}{\partial x_2}(P) & \cdots & \dfrac{\partial f_m}{\partial x_n}(P) \end{bmatrix}.$$

Hence multiplication by this matrix induces a short exact sequence

$$\{0\} \longrightarrow T_P(X) \longrightarrow \mathbb{A}^n(k) \longrightarrow \mathbb{A}^m(k)$$

where $T_P(X)$ is the *tangent space of X at P*. We have a perfect (i.e., bilinear and nondegenerate) pairing $T_P(X) \times \big(\mathfrak{m}_P/\mathfrak{m}_P^2\big) \to k$ defined by $\big(Q,\, f\big) \mapsto \nabla f(P) \cdot Q$. (For this reason, we call $\mathfrak{m}_P/\mathfrak{m}_P^2$ the *cotangent space of X at P*.) Hence $\dim_k\big(\mathfrak{m}_P/\mathfrak{m}_P^2\big) = \dim_k T_P(X)$. But $\dim_k T_P(X) = n - m$ if and only if $J\big(f_1,\, f_2,\, \ldots,\, f_m\big)(P)$ has rank m. The case of interest follows with $n = 2$ and $m = 1$.

(ii) $\iff$ (iii). We will abuse notation and think of $\mathfrak{m}_P$ as an ideal of $\mathcal{O}_P$. As $\mathfrak{m}_P$ is a maximal ideal, Nakayama's Lemma states that we can find $t \in \mathfrak{m}_P$ where $t \notin \mathfrak{m}_P^2$. Consider the injective map $\mathcal{O}/\mathfrak{m}_P \to \mathfrak{m}_P/\mathfrak{m}_P^2$ defined by $x \mapsto t\,x$. Clearly this is surjective if and only if $\mathfrak{m}_P = t\,\mathcal{O}_P$ is principal. Recall now that $\dim_k\big(\mathcal{O}/\mathfrak{m}_P\big) = 1$.

(iii) $\implies$ (iv). Say that $\mathfrak{m}_P = t\,\mathcal{O}_P$ as a principal ideal. In order to show that $\mathcal{O}_P$ is a discrete valuation ring, it suffices to show that any nonzero $x \in \mathcal{O}_P$ is in the form $x = t^m\,y$ for some $m \in \mathbb{Z}$ and $y \in \mathcal{O}_P^\times$. Consider the radical of the ideal generated by x:

$$\sqrt{(x)} = \big\{ y \in \mathcal{O}_P \,\big|\, y^n \in x\,\mathcal{O}_P \text{ for some nonnegative integer } n \big\}.$$

As $\mathcal{O}_P$ has a unique nonzero prime ideal, we must have $\sqrt{(x)} = \mathfrak{m}$. But then there is largest nonnegative integer m such that $t^{m-1} \notin x\,\mathcal{O}_P$ yet $t^m \in x\,\mathcal{O}_P$. Hence $y = x/t^m \in \mathcal{O}_P$ but $y \notin \mathfrak{m}_P$.

(iv) $\implies$ (v). Say that $\mathcal{O}_P$ is a discrete valuation ring. Say that $x \in K$ is a root of a polynomial equation $x^n + a_1\,x^{n-1} + \cdots + a_n = 0$ for some $a_i \in \mathcal{O}_P$. Assume by way of contradiction that $x \notin \mathcal{O}_P$. Then $v_P(x) < 0$, so that $v_P(1/x) > 0$, hence $y = 1/x$ is an element of $\mathcal{O}_P$. Upon dividing by x^{n-1} we have the relation $x = -\big(a_1 + a_2\,y + \cdots + a_n\,y^{n-1}\big) \in \mathcal{O}_P$. This contradiction shows that $\mathcal{O}_P$ is indeed integrally closed.

(v) $\implies$ (iii). Say that $\mathcal{O}_P$ is integrally closed. We must construct an element $t \in \mathcal{O}_P$ such that $\mathfrak{m}_P = t\,\mathcal{O}_P$. Fix a nonzero $x \in \mathfrak{m}_P$. By considering the radical $\sqrt{(x)}$ and noting that $\mathfrak{m}_P$ is a finitely generated $\mathcal{O}_P$-module, we see that there exists some $m \in \mathbb{Z}$ such that $\mathfrak{m}_P^m \subseteq x\,\mathcal{O}_P$ yet $\mathfrak{m}_P^{m-1} \not\subseteq x\,\mathcal{O}_P$. Choose $y \in \mathfrak{m}_P^{m-1}$ such that $y \notin x\,\mathcal{O}_P$, and let $t = x/y$ be an element in K. Consider the module $(1/t)\,\mathfrak{m}_P \subseteq \mathcal{O}_P$; we will show equality. As $y \notin \mathcal{O}_P$, we have $1/t \notin \mathcal{O}_P$, so that $1/t$ is not integral over $\mathcal{O}_P$. Then $(1/t)\,\mathfrak{m}_P$ cannot be a finitely generated $\mathcal{O}_P$-module, we have $(1/t)\,\mathfrak{m}_P \not\subseteq \mathfrak{m}_P$. As there is an element of $(1/t)\,\mathfrak{m}_P$ which is not in $\mathfrak{m}_P$, we must have equality: $(1/t)\,\mathfrak{m}_P = \mathcal{O}_P$. Hence $\mathfrak{m}_P = t\,\mathcal{O}_P$ as desired. □

Examples. Take $f_1(x,y) = y^2 - x^3 + x$, and $P = (0,0)$. Then $\nabla f_1(P) = (1,0)$ is nonzero. The maximal ideal is $\mathfrak{m}_P = \big(x,\, y\big)$ and $\mathfrak{m}_P^2 = \big(x^2,\, x\,y,\, y^2\big)$, so that $x = x \cdot x^2 + (-1) \cdot y^2 \in \mathfrak{m}_P^2$. Hence $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a 1-dimensional k-vector space spanned by y alone. Now take $f_1(x,y) = y^2 - x^3$, and $P = (0,0)$. Then $\nabla f_1(P) = (0,0)$ is zero. The quotient $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a 2-dimensional k-vector space spanned by both x and y.

Dedekind Domains. Here is an application to the rings one studies in Algebraic Number Theory.

Corollary 3. *The following are equivalent:*

i. X *is a nonsingular curve, i.e.,* $\nabla f_1(P) \neq (0,0)$ *for all points* $P \in X$.
ii. $\dim_k\big(\mathfrak{m}/\mathfrak{m}^2\big) = 1$ *for all maximal ideals* $\mathfrak{m}$ *of* $\mathcal{O}$.
iii. $\mathcal{O}$ *is a Dedekind domain.*

Proof. (i) $\iff$ (ii). Recall that the map $X \to \operatorname{mSpec} \mathcal{O}$ defined by $P \mapsto \mathfrak{m}_P$ is a bijection. Now use the previous theorem. (ii) $\iff$ (iii). A Dedekind domain is a Noetherian integral domain of dimension 1 that is integrally closed. Using the previous theorem, it suffices show that the localization $\mathcal{O}_{\mathfrak{m}}$ is integrally closed for each maximal ideal $\mathfrak{m}$ if and only if $\mathcal{O}$ is integrally closed. But this is clear. (If not, consult Theorem 5.13 on page 63 of Atiyah-Macdonald.) □

Abstract Nonsingular Curves. Let $\mathcal{O}$ be a Noetherian integral domain with Krull dimension $\dim \mathcal{O} = 1$. We say that $X = \operatorname{Spec} \mathcal{O}$ is an *abstract curve.* Note that X is *nonsingular* precisely when $\mathcal{O}$ is a Dedekind domain. For instance, when $\mathcal{O} = \mathbb{Z}$, we think of $X = \operatorname{Spec} \mathbb{Z}$ as a *nonsingular* curve!

Abstract Nonsingular Varieties. This can be generalized considerably. Let $\mathcal{O}$ be a Noetherian integral domain. As $\mathcal{O}$ is an integral domain, we have $\mathcal{O} \subseteq K$; as $\mathcal{O}$ is Noetherian, the Krull dimension, $\dim \mathcal{O}$, is finite. We say that the abstract variety $X = \operatorname{Spec} \mathcal{O}$ is *nonsingular* if $\mathcal{O}$ is integrally closed. It is not hard to show that $\dim_k\big(\mathfrak{m}/\mathfrak{m}^2\big) = \dim \mathcal{O}$ for *any* maximal ideal $\mathfrak{m}$ of $\mathcal{O}$; we define this as the dimension of X.