Involution

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Let T be a torus with one conic singularity $a \in T$, with a metric of constant curvature 1*. Then there exists an isometric involution which fixes a.*

Let f be the developing map, then the Schwarzian of f is a single valued doubly periodic meromorphic function in the plane, with poles of multiplicity 2 at the lattice points. So $f = w_1/w_2$, the ratio of two solutions of the equation

$$w'' - (k\wp(z) + \lambda)w = 0, \tag{1}$$

where k is he positive constant related to the angle at the singularity, and λ is the accessory parameter.

The indicial equation is

$$\rho(\rho - 1) - k = 0, \tag{2}$$

so the exponents are $\rho_{1,2} = 1/2 \pm \sqrt{1/4 + k}$, and the exponent difference is $\alpha = 2\sqrt{1/4 + k}$. This difference is the angle at the singularity (measured in turns), so $k = (\alpha^2 - 1)/4$.

We rewrite the equation as

$$z^2 w'' - w \sum_{j=0}^{\infty} a_j z^j = 0,$$

where $a_0 = k,\ a_2 = \lambda$, and $a_{2j+1} = 0$ for $j \geq 0$, since $\wp$ is even. If α is not an integer, then the equation has two solutions

$$w_i(z) = z^{\rho_i} \sum_{j=0}^{\infty} c_{i,j} z^j, \quad i = 1, 2.$$

Substituting these series to the equation we obtain (suppressing the index i)

$$\sum_{n=0}^{\infty}(n+\rho)(n+\rho-1)c_n z^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_n c_{n-m}\right) z^n.$$

Then for c_0 we have

$$\rho(\rho-1)c_0 = kc_0,$$

so c_0 is arbitrary in view of (2), and for c_n we have

$$(n+\rho)(n+\rho-1)c_n = \sum_{m=0}^{n} a_m c_{n-m}.$$

As all a_j with odd subscript are 0, e obtain that $c_1 = 0$, and then by induction we obtain that all c_j with odd subscript are zero. Thus

$$w_i(z) = z^{\rho_i} g_i(z),$$

where g_i are even functions holomorphic in a neighborhood of the origin. Then the developing map $f(z) = w_1(z)/w_2(z) = z^{\alpha}h(z)$, where h is an even function, and the density of the metric

$$2\frac{|f'|}{1+|f|^2}$$

is even.

When α is an odd integer, both w_i are meromorphic, one of them is even another odd, so f is odd and the density of the metric is even.

When α is an even integer, we have a condirion on λ which ensures that there is no logarithm, and when this condition is satisfied, f is also odd and the density of the metric is even.