$$A_m = \sum_{2^k \leq m+1} \left(1 - \frac{1}{2^k}\right) \cdot \alpha(2^k - 1, m).$$

6.9. From here there follows at once that $0 < A_m < A_{m+1}$.

The computer calculations allow us to presuppose that in the considered case (p = 2) we have, asymptotically,

$$A_m \sim \text{const} \cdot 2^m \cdot m^{-\gamma};\ B_m \sim \text{const} \cdot 2^m \cdot m^{-1-\gamma},\ \gamma \approx 0{,}48.$$

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A NEW PROOF OF DRASIN'S THEOREM ON MEROMORPHIC FUNCTIONS OF FINITE ORDER WITH MAXIMAL DEFICIENCY SUM. I

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1. Introduction. For a function f, meromorphic in the plane $\mathbb{C}$, we make use of the standard notations of the R. Nevanlinna theory: T(r, f), $N(r, a)$, $m(r, a)$, $\bar{N}(r, f)$, $N_1(r)$, $\delta(a)$. In addition, we set $D(z_0, R) = \{z \in \mathbb{C}: |z - z_0| < R\}$. In this paper we investigate meromorphic functions of finite lower order with maximal deficiency sum:

$$\sum_{a \in \bar{\mathbb{C}}} \delta(a) = 2. \tag{1.1}$$

For a function f of finite order, R. Nevanlinna's second fundamental theorem can be formulated in the following form: for each finite collection $a_1, \ldots, a_q$ we have

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leq 2T(r, f) + o(T(r, f)),\ r \to \infty.$$

From here and from (1.1) there follows that

$$N_1(r) = o(T(r, f)),\ r \to \infty. \tag{1.2}$$

In order to elucidate what consequences can (1.1) imply, we assume first that a stronger condition than (1.2) is satisfied, namely, $N_1(r) \equiv 0$, i.e., f does not have multiple points. We consider the Schwarzian derivative

$$F = f'''/f' - (3/2)(f''/f')^2. \tag{1.3}$$

A simple computation shows that the Schwarzian derivative has poles only at the multiple points of the function f and, therefore, F is an entire function. Taking into account that f is of

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finite order, with the aid of the lemma on the logarithmic derivative we obtain that $m(r, F) = O(\log r)$, $r \to \infty$, and, consequently, F is a polynomial. Now (1.3) can be considered as an algebraic differential equation with respect to f. The general solution of this equation represents the ratio of two linearly independent solutions of the linear equation $y'' + \frac{1}{2}Fy = 0$. In 1932, making use of this circumstance, R. Nevanlinna has investigated in a detailed manner meromorphic functions of finite order, without multiple points. These functions have the following properties:

a) $T(r, f) \sim cr^{n/2}$, where $c > 0$, $n \geq 2$ is a natural number;

b) the plane is partitioned into n equal angular domains: $D_j = \{z : \varphi_{j-1} < \arg z < \varphi_j\}$, $1 \leq j \leq n$, $\varphi_n = \varphi_0$ so that for some numbers $b_j \in \bar{\mathbb{C}}$ we have

$$\log \frac{1}{|f(re^{i\varphi}) - b_j|} = \pi c r^{n/2} \sin \frac{n}{2}(\varphi - \varphi_{j-1}) + o(r^{n/2}),$$

when $r \to \infty$, uniformly with respect to ϕ in any angle that lies strictly inside D_j. If $b_j = \infty$, then the left-hand side has to be replaced by $\log|f(re^{i\varphi})|$.

Thus, if the number $a \in \bar{\mathbb{C}}$ occurs among the numbers b_j $p(a)$ times, then $\delta(a) = 2p(a)/n$. All the deficiency values are asymptotic.

Another approach for obtaining the given result, due to L. Ahlfors, consists in the investigation of the Riemann surface onto which the function f maps the plane. It can be shown that this Riemann surface has a finite number of logarithmic branching points and does not have algebraic branching points. Such Riemann surfaces admit a comprehensive description and the assertions a) and b) are obtained with the aid of an explicit construction of a mapping of the Riemann surface onto the plane, close to a conformal one.

The presented arguments lead in a natural manner to a conjecture, stated for the first time in 1929 by F. Nevanlinna. Let f be a meromorphic function of finite order ρ, possessing property (1.1). Then the following statements hold:

1) 2ρ is a natural number ≥ 2.

2) If $\delta(a) > 0$, then $\delta(a) = p(a)/\rho$, where $p(a)$ is a natural number.

3) All the deficiency values are asymptotic.

From 2) there follows that the number of deficiency values does not exceed 2ρ.

For entire functions this conjecture has been proved in 1946 by A. Pfluger. In this case, statement 1) can be refined: ρ is a natural number. The first substantial headway in F. Nevanlinna's conjecture for meromorphic functions was A. Weitsman's result in 1969: under the conditions of the conjecture, the number of deficiency values does not exceed $2\rho_1$, where ρ_1 is the lower order, $\rho_1 \leq \rho$. After a series of intermediate results, a complete proof of the statements 1), 2), 3) has been obtained recently by D. Drasin.* It is one of the longest and most complex proof in function theory. D. Drasin's proof makes use of a series of miscellaneous auxiliary means, like Ahlfors' theory of covering surfaces of quasiconformal mappings.

In this paper we present a new proof, based on two fundamental theorems of the R. Nevanlinna theory and of classical potential theory. The author hopes that this proof makes D. Drasin's remarkable result more accessible and that the presented method will find further applications. Incidentally, we shall prove the above formulated theorem of A. Weitsman.

THEOREM 1. Let f be a meromorphic function of finite lower order and having property (1.1). Then statements 1), 2), 3) hold. If, in addition, $\delta(\infty) = 0$, then we have

$$\log \frac{1}{|f'(re^{i\theta})|} = \pi r^{\rho} l_1(r) |\cos \rho(\theta - l_2(r))| + o(r^{\rho} l_1(r)), \tag{1.4}$$

uniformly with respect to θ for $r \to \infty$, $re^{i\theta} \notin C_0$. Here C_0 is the union of the circles $D(z_k, r_k)$ such that

$$\sum_{\{k : |z_k| < R\}} r_k = o(R), \ R \to \infty,$$

*D. Drasin, Proof of a conjecture of F. Nevanlinna concerning functions which have deficiency sum two, Acta Math., Vol. 158, No. 1-2, pp. 1-94 (1987).

while ℓ_j are continuous functions with the properties $l_1(ct) \sim l_1(t)$, $l_2(ct) = l_2(t) + o(1)$, $t \to \infty$ uniformly with respect to $c \in [1, 2]$.

In addition,

$$T(r, f) \sim r^{\rho} l_1(r), \; r \to \infty. \tag{1.5}$$

Conversely, each meromorphic function, having the properties (1.4), (1.5) (2ρ is a natural number), satisfies relation (1.1).

The above given arguments on meromorphic functions of finite order with the property $N_1(r) \equiv 0$ allow us to presuppose that Theorem 1 remains valid if in its assumptions we replace (1.1) by (1.2). Such a refinement of Theorem 1 remains unproved.

2. The Definition of the Functions u, u_j. We denote by L^1_{loc} the space of functions that are summable on each circle in $\mathbb{C}$. The subharmonic functions are contained in L^1_{loc}. Let v_1, v_2 be subharmonic functions. The element $v = v_1 - v_2 \in L^1_{loc}$ is called a δ-subharmonic function. The "function" v may be undefined at those points where $v_1 = v_2 = -\infty$. We say that a δ-subharmonic function v is defined at the point z if there exists a finite or infinite limit

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_0^{2\pi} d\theta \int_0^r v(z + te^{i\theta})\, t\, dt,$$

and we shall denote this limit by $v(z)$. The definition is correct since for a subharmonic function v the indicated limit coincides with $v(z)$. Obviously, if a δ-subharmonic function $v \geq 0$ a.e., then $v(z) \geq 0$ at all the points z where v is defined. In this case we write simply $v \geq 0$.

We proceed to the proof of Theorem 1. The scheme of the proof is the following. In Secs. 2-6 the theorem will be reduced to a statement in potential theory, which will be called Fundamental Lemma (see Sec. 6). Accepting the fundamental lemma, we prove Theorem 1 in Sec. 7. The proof of the fundamental Lemma, independent of everything else, is contained in the second part of the papers (Secs. 8-11).

Without loss of generality, we can assume that all the poles of the function f are simple and that we have $\bar{N}(r, f) = N(r, f) \sim T(r, f), r \to \infty$ (2.1). From here $m(r, f) = o(T(r, f)), r \to \infty$ (2.2). All this can be achieved by performing on f a linear fractional transformation. In this case the finiteness of the lower order and condition (1.1) are preserved.

We recall that a sequence $r_m \to \infty$ is called a sequence of Pólya peaks of order λ of the increasing function $T(r)$ if for some sequence $\varepsilon_m \to 0$ we have

$$T(r) \leqslant (1 + \varepsilon_m)\left(\frac{r}{r_m}\right)^{\lambda} T(r_m), \; \varepsilon_m r_m \leqslant r \leqslant \frac{r_m}{\varepsilon_m}. \tag{2.3}$$

We set

$$\rho^* = \sup\left\{p : \limsup_{x,\, A \to \infty} \frac{T(Ax)}{A^p T(x)} = \infty\right\};$$
$$\rho_1^* = \inf\left\{p : \liminf_{x,\, A \to \infty} \frac{T(Ax)}{A^p T(x)} = 0\right\}.$$

It is known† that Pólya peaks of order λ exist if and only if $\rho_1^* \leq \lambda \leq \rho^*$. In addition, $[\rho_1, \rho] \subset [\rho_1^*, \rho^*]$, where ρ_1, ρ are the order and the lower order, respectively, of the function $T(r)$. We fix a number $\lambda \in [\rho_1^*, \rho^*]$, $\lambda < \infty$, and a sequence of Polya peaks r_m for the function $T(r) = T(r, f)$. In the course of the proof we shall select several times a subsequence from the sequence r_m, preserving for it the previous notation. According to R. Nevanlinna's second fundamental theorem, for each finite collection $\{a_1, \ldots, a_q\} \subset \bar{\mathbb{C}}$ we have

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leqslant 2T(r) + o(T(2r)), \; m \to \infty \tag{2.4}$$

†D. Drasin and D. F. Shea, Pólya peaks and the oscillation of positive functions, Proc. Am. Math. Soc., Vol. 34, No. 2, pp. 403-411 (1972).

(we write the remainder in this form since the finiteness of the order of the function f is not assumed a priori and we need a relation without the exceptional set). From (1.1), (2.4), (2.3) there follows that for each $t > 0$ we have

$$N_1(tr_m) = o(T(r_m)),\ n_1(tr_m) = o(T(r_m)),\ m \to \infty. \tag{2.5}$$

Let a_j, $j = 1, 2, \ldots$, be all the deficiency values of the function f (we do not assume that their set is finite). We conisder the δ-subharmonic functions:

$$\begin{gathered} U_m(z) = (\log|f'(zr_m)|^{-1})/T(r_m), \\ U_{m,j}(z) = (\log|f(zr_m) - a_j|^{-1})/T(r_m). \end{gathered} \tag{2.6}$$

We make use of the following result, due to J. M. Anderson — A. Baernstein II* and V. S. Azarin†: from condition (2.3) there follows that the families $\{U_m\}$ and $\{U_{mj}\}$ are relatively compact in the following sense. One can select a sequence of Pólya peaks so that we have $U_m \to u,\ U_{mj} \to u_j,\ m \to \infty$ (2.7). Here u and u_j are some δ-subharmonic functions. The convergence in (2.7) takes place in L^1_{loc} and also in L^1 on each circumference. The Riesz charges of the functions U_m and U_{mj} converge weakly to the Riesz charges of the functions u and u_j, respectively. By the 1-measure of some set $E \subset \mathbb{C}$ we mean the greatest lower bound of the sums of the radii of the circles that cover E. For each circle and for each $\varepsilon > 0$ the subsequence of Pólya peaks can be chosen so that the convergence in (2.7) be uniform in this circle, outside some set whose 1-measure is less than ε. Regarding these results, see also [1, 2].

From $\delta(a_j, f) > 0$ there follows that $u_j \not\equiv 0$, $j = 1, 2, \ldots$.

The functions u and u_j play a fundamental role in the proof. In Secs. 3-6 the assumptions of the theorem will be reformulated in terms of u and u_j and we obtain the fundamental lemma from Sec. 6, which is the "subharmonic analogue" of Theorem 1. From the fundamental Lemma it will follow that

$$\sum_j u_j = u = \pi r^{\lambda} |\cos\lambda(\theta - \theta_0)|,$$

where $\theta_0 \in [-\pi, \pi]$ and 2λ is a natural number. Reformulating this statement in terms of the function f, we obtain (1.4) and then all the remaining assertions of Theorem 1 (Sec. 7).

From (2.1), (2.5) there follows that

$$m\left(tr_m, \frac{1}{f'}\right) \sim T\left(tr_m, \frac{1}{f'}\right) \sim 2T(tr_m, f),\ m \to \infty$$

for any $t > 0$. Taking into account (2.3) and taking the limit as $m \to \infty$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\theta})\,d\theta \leqslant 2r^{\lambda},\ 0 < r < \infty, \tag{2.8}$$

moreover, for $r = 1$ we have equality in (2.8).

3. The Simplest Properties of the Functions u and u_j. We make use of the lemma on the logarithmic derivative in the following form:

$$m(r, f'/f) = o(T(2r)),\ r \to \infty. \tag{3.1}$$

From (2.2), (3.1) there follows that $m(r, f') = o(T(2r))$, $r \to \infty$. Taking into account (2.3) and taking the limit in L^1 on circumferences, we obtain $u \geq 0$, $u_j \geq 0$, $j = 1, 2, \ldots$ (3.2). Further, from (2.5) there follows that u is a subharmonic function, in particular, u is defined everywhere in $\mathbb{C}$. From the lemma on the logarithmic derivative, applied to the functions

*J. M. Anderson and A. Baernstein II. The size of the set on which a meromorphic function is large, Proc. London Math. Soc. 36, 518-539 (1978).
†V. S. Azarin, On the asymptotic behavior of subharmonic functions of finite order, Mat. Sb., Vol. 108 (150), No. 2, pp. 147-167 (1979).

$f - a_j$, there follows that $u \geq u_j$, $j = 1, 2, \ldots$ (3.3) in the domain of definition of the function u_j.

We fix j and we consider all possible closed Jordan polygons Γ, on which the function u_j is defined and $\inf\{u_j(z): z \in \Gamma\} > 0$. We denote by D_j the union of the interior domains of all such polygons. Obviously, the set D_j is open and all its connected components are simply connected.

We show that if $u_j(z_0) > 0$, then $z_0 \in D_j$. Let $u_j = v_1 - v_2$, v_i are subharmonic functions, $u_j(z_0) = d > 0$. By virtue of upper semicontinuity, we have $v_2(z) < v_2(z_0) + d/3$ in some neighborhood V of the point z_0. From the well-known properties of potentials [3, Chap. VII, Sec. 5, Corollary] there follows that there exists a square contour $\Gamma \subset V$, surrounding the point z_0 such that $v_1(z) > v_1(z_0) - d/3$, $z \in \Gamma$. Therefore, $u_j(z) \geqslant u_j(z_0) - 2d/3 > d/3 > 0$, $z \in \Gamma$, and $z_0 \in D_j$.

4. Proof of the Fact that the Sets D_j are Pairwise Disjoint. Assume, for example, that $D_1 \cap D_2 \neq \varnothing$. Then there exist simple closed polygons Γ_1, Γ_2, whose interior domains intersect and, moreover, $u_1(z) > d$, $z \in \Gamma_1$; $u_2(z) > d$, $z \in \Gamma_2$; $d > 0$. Since $a_1 \neq a_2$ and the convergence in (2.7) is uniform on $\Gamma_1 \cup \Gamma_2$ outside a set of small linear measure, we have $\Gamma_1 \neq \Gamma_2$. Then one of the polygons (Γ_1, say) contains a point z_0, lying in the domain bounded by the polygon Γ_2. From (3.3) there follows that $u(z_0) > d$. From the maximum principle, applied to the subharmonic function u, and from the upper semicontinuity of this function there follows that there exists a continuum E such that $u(z) \geqslant d$, $z \in E$; $z_0 \in E$, $E \cap \Gamma_2 \neq \varnothing$. Now we make use of the following lemma.

LEMMA 1. Let v be a subharmonic function, $v(0) = d > 0$. Then there exists a natural number N such that for any $n \geq N$ the set of the values of r from the interval $(2^{-n-1}, 2^{-n})$ such that $v(re^{i\theta}) > \frac{d}{2}$, $|\theta| \leqslant \pi$, has length $\geq 2^{-n-2}$.

Proof. Let $K = \{z: v(z) < d/2\}$. The set K is thin at zero by the definition of thinness [3, 4]. Consequently, the circular projection of the set K onto the positive ray is thin at zero [4, Proposition IX.2] and the lemma follows from N. Wiener's thinness criterion [4, Theorem IX.10].

Let $R > 0$ be so large that $E \subset D(0, R/2)$. For each $z \in E$ we select a number N(z) so that the assertion of Lemma 1 should hold with the point z instead of the point 0 and with the function u for v. In addition, we assume that

$$2^{-N(z)} < \min\{\operatorname{diam} \Gamma_1, \operatorname{diam} \Gamma_2\}. \tag{4.1}$$

There exists a set X(z) with 1-measure not exceeding $2^{-N(z)-2}$ and such that the convergence in (2.7) with j = 1, 2 is uniform on the set $D(0, R)\backslash X(z)$. If necessary, we select a subsequence in (2.7). Making use of Lemma 1, we find a circumference C(z) with center at the point z such that $u(\zeta) > d/2$, $\zeta \in C(z)$, and the convergence in (2.7) with j = 1, 2 is uniform on C(z). We select the radius of this circumference C(z) so that it should not exceed $2^{-N(z)}$. Let D(z), $z \in E$, be the circles bounded by the circumferences C(z). One can select a finite covering of the set E by these circles so that no circle of the covering be contained entirely in another circle of the covering. From the arcs of the circumferences of the selected circles one can form a rectifiable curve Γ, possessing the following properties: $u(z) > d/2$, $z \in \Gamma$ (4.2), the endpoints z_1 and z_2 of the curve Γ belong to Γ_1 and Γ_2, respectively (this can be achieved by virtue of (4.1)); the limits in (2.7) for j = 1, 2 are uniform on Γ.

Let $r_m\Gamma = \{z: z/r_m \in \Gamma\}$. From (4.2) and from the uniform convergence in (2.7) there follows that $|f'(z)| \leqslant \exp(-cT(r_m))$, $z \in r_m\Gamma$ with some constant $c > 0$. Taking into account that the length of the curve $r_m\Gamma$ is $O(r)$, $m \to \infty$, we integrate along the curve $r_m\Gamma$ and we obtain that $|f(r_m z_1) - f(r_m z_2)| = O(r_m \exp(-cT(r_m))) = o(1)$, $m \to \infty$. This contradicts the fact that $f(r_m z_j) \to a_j$, $m \to \infty$, $j = 1, 2$. We have proved that $D_i \cap D_j = \varnothing$ $i \neq j$.

5. Proof of A. Weitsman's Theorem. We show that $u(z) = 0$ for $z \in \partial D_j$, $j \in \mathbb{N}$. Assume, for example $u(z_0) = d > 0$, $z_0 \in \partial D_1$. Making use of Lemma 1, we find a sufficiently small circumference $C(z_0)$ such that $u(z) > d/2$, $z \in C(z_0)$, and the convergence in (2.7) is uniform on $C(z_0)$. From the definition of D_1 there follows that there exists a point $z_1 \in C(z_0)$ such that $u_1(z_1) > 0$. Reasoning as above in Sec. 4, we obtain that $u_1(z) > 0$ for $z \in C(z_0)$; contradiction.

Now we note that u is a subharmonic function of finite order ($\leq\lambda$). This follows from (2.8). Each connected component D_{jk} of the set D_j contains at least one connected component of the set $\{z: u(z) \geqslant \varepsilon_{jk} > 0\}$. From here it follows that the set of such components D_{jk} is finite ($\leq \max\{1, 2\lambda\}$) [5, Theorem 4.16].

We note that so far we have used only (1.2) and not the stronger condition (1.1). Thus, we have proved a certain generalization of A. Weitsman's theorem: functions of finite lower order, having property (1.2), have a finite set of deficiency values. We denote the number of deficiency values by q.

6. A Subharmonic Analogue of Theorem 1. By the support of a δ-subharmonic function we mean the set where it is defined and different from 0. From the results of Secs. 3, 4 there follows that the supports of the functions u_j are pairwise disjoint. Therefore,

$$\sum_{j=1}^{q} u_j = \max_{1 \leqslant j \leqslant q} u_j \text{ a.e.}$$

and from (3.3) we obtain

$$u(z) \geqslant \sum_{j=1}^{q} u_j(z) \tag{6.1}$$

where the right-hand side is defined. Now we make use of condition (1.1). Taking into account (2.1) and (3.1), for each $r > 0$ we have

$$\sum_{j=1}^{q} m(rr_m, a_j) \sim 2T(rr_m, f) \sim T(rr_m, f') \sim m\left(rr_m, \frac{1}{f'}\right), \ m \to \infty.$$

From here and from (2.7) there follows that

$$\int_0^{2\pi} \left\{\sum_{j=1}^{q} u_j(re^{i\theta})\right\} d\theta = \int_0^{2\pi} u(re^{i\theta})\, d\theta, \ r > 0.$$

Together with (6.1) this yields

$$\sum_{j=1}^{q} u_j(z) = u(z), \ z \in \mathbf{C}. \tag{6.2}$$

We show that the functions u_j are subharmonic. Indeed, u_j is subharmonic in D_j since from (6.2) and the fact that $u_k = 0$ in D_j for $k \neq j$ there follows that $u_j = u$ in D_j. Further, $u_j = 0$ on ∂D_j because $0 \leq u_j \leq u$ everywhere and $u = 0$ on ∂D_j. In addition, $u_j = 0$ outside D_j. Therefore, u_j are subharmonic functions.

We denote by μ (by μ_j) the measure associated according to Riesz with the function u (with the function u_j). From (6.2) there follows the relation

$$\mu = \sum_{j=1}^{q} \mu_j. \tag{6.3}$$

We denote by ν the measure counting the poles of the function f. (This means that ν(E) is the number of poles in the Borel set E.) By ν_j we denote the measure counting the a_j-points. For any measure τ we denote by $(\tau)_t$ the measure defined in the following manner: $(\tau)_t(E) = \tau(tE)$, $t > 0$, E is any Borel set. From (2.7) there follows the weak convergence of the corresponding Riesz charges:

$$(\nu)_{r_m}/T(r_m) \to \frac{1}{2}\mu,$$

$$((\nu)_{r_m} - (\nu_j)_{r_m})/T(r_m) \to \mu_j,$$

from where we obtain that $\frac{1}{2}\mu \geq \mu_j$, $1 \leq j \leq q$ (6.4). From (6.3), (6.4) we obtain that

$$\sum_{j=1}^{q} \mu_j \geqslant 2\mu_k, \ 1 \leqslant k \leqslant q. \tag{6.5}$$

Fundamental Lemma. Let D_j be pairwise disjoint open sets, consisting of a finite number of simply connected domains, and let $u_j \not\equiv 0$ be nonnegative subharmonic functions, whose supports are contained in D_j, respectively. Assume that the Riesz measures μ_j of these functions satisfy condition (6.5) and, in addition, we have

$$\frac{1}{2\pi}\sum_{j=1}^{q}\int_0^{2\pi} u_j(re^{i\theta})\,d\theta \begin{cases} \leqslant 2r^{\lambda+\varepsilon}, & r_0 \leqslant r < \infty, \\ = 2, & r = 1, \\ \leqslant 2r^{\lambda-\varepsilon}, & 0 \leqslant r \leqslant r_0^{-1}, \end{cases} \tag{6.7}$$

where $0 \leqslant \varepsilon < \frac{1}{4}$, $r_0 > 1$ are some numbers. Then there exists an integer $n \geqslant 2$, $|n/2 - \lambda| < 1/2$ such that

$$u(re^{i\theta}) = \sum_{j=1}^{q} u_j(re^{i\theta}) = \pi r^{n/2}\left|\cos\frac{n}{2}(\theta-\theta_0)\right| \tag{6.8}$$

for some θ_0, $0 \le \theta_0 < 2\pi$.

The proof of the Fundamental Lemma is contained in the second part of this work.

7. The Conclusion of the Proof of Theroem 1. We verify that the conditions of the Fundamental Lemma hold. The fact that the sets D_j are pairwise disjoint is proved in Sec. 4; the fact that they consist of a finite number of domains is proved in Sec. 5; relation (6.5) has been proved in Sec. 6. Finally, from (2.8), (6.2) there follows (6.7) with $\varepsilon = 0$.

Applying the Fundamental Lemma, we obtain (6.8). We have proved the following

Statement 1. Assume that the meromorphic function satisfies condition (1.1) and that for some sequence $r_m \to \infty$ the condition (2.3) holds. We define U_m, U_{mj} by the formulas (2.6). Then for some subsequence of the indices m we have $U_m \to u$, $U_{mj} \to u_j$, where u and u_j have the form (6.8).

From the comparison of (6.8) and (2.8) there follows that $\lambda = n/2$. Thus, all the possible orders λ of the Polya peaks are semiintegers. On the other hand, as indicated in Sec. 2, the possible orders of the Polya peaks fill out the segment $[\rho_1^*, \rho^*]$, containing the segment $[\rho_1, \rho]$. Consequently, $\rho_1^* = \rho^* = \rho_1 = \rho = n/2$ and, in particular, we have proved that the function f has a finite order and we have established the validity of statement 1) of Sec. 1. Since $\rho_1^* = \rho^*$, from the formulas for ρ_1^*, ρ^*, given in Sec. 2, there follows that for each $\varepsilon > 0$ there exist $r_0 > 1$, $x_0 > 1$ such that

$$T(tx) \leqslant t^{\rho+\varepsilon}T(x), \quad t > r_0,\ x > x_0; \tag{7.1}$$

$$T(tx) \leqslant t^{\rho-\varepsilon}T(x), \quad t < r_0^{-1},\ tx > x_0. \tag{7.2}$$

These relations are sufficient in order to replace (2.3) in Statement 1; i.e., we have

Statement 2. Assume that the meromorphic function f satisfies the conditions (1.1), (7.1), (7.2) with $\rho = n/2$, n being a natural number, $n \ge 2$. For an arbitrary sequence $r_m \to \infty$ we define U_m, U_{mj} by the formulas (2.6). Then for some subsequence of the indices m we have $U_m \to u$, $U_{mj} \to u_j$, where u and u_j are functions of the form (6.8).

Indeed, the conditions (7.1), (7.2) with $x = r_m$ ensure the applicability of the theorems of J. M. Anderson and A. Baernstein II and of V. S. Azarin on the compactness of the sequences U_m, U_{mj}. Selecting subsequences, we obtain (2.7). Instead of (2.8), by a limiting process, from (7.1), (7.2) with $x = r_m$ we obtain the relation

$$\frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\theta})\,d\theta \leqslant \begin{cases} 2r^{\rho+\varepsilon}, & r \geqslant r_0, \\ 2r^{\rho-\varepsilon}, & r \leqslant r_0^{-1} \end{cases} \tag{7.3}$$

with equality for $r = 1$. In the sequel, Statement 2 is proved in the same way as Statement 1, but with the following modifications. For the estimations of the remainders in (2.4), (3.1), instead of (2.3) we make use of (7.1). In order to prove that the sets D_j consist of a finite number of domains, in Sec. 5 instead of (2.8) we make use of (7.3). Finally, for the verification of the condition (6.7), instead of (2.8) we make use of (7.3).

From Statement 2 and from (2.5) there follows that $T(cr)/T(r) \to c^\rho$, $r \to \infty$ uniformly with respect to $c \in [1, 2]$. Setting $T(r) = r^\rho \ell_1(r)$, we obtain $\ell_1(cr) \sim \ell_1(r)$, $r \to \infty$ uniformly with respect to $c \in [1, 2]$, i.e., (1.5) holds.

We prove (1.4). We denote by X the set consisting of the subharmonic functions of the form

$$u(re^{i\theta};\ \theta_0) = \pi r^\rho |\cos\rho(\theta - \theta_0)|,\ \theta_0 \in [-\pi,\ \pi).$$

Obviously, the set X is compact in L^1_{loc}. We note that L^1_{loc} is a metric space. We consider the family of functions

$$v_t(z) = \left(\log\frac{1}{|f'(zt)|}\right)\Big/(t^\rho l_1(t)).$$

We show that $\mathrm{dist}(v_t, X) \to 0$, $t \to \infty$ (7.4). Assume that (7.4) does not hold. Then there exists a sequence $t_m \to \infty$ such that $\mathrm{dist}(v_{t_m}, X) \geq \varepsilon > 0$, $m \to \infty$. Taking this sequence for r_m, we apply Statement 2. We obtain that for some subsequence $v_{t_m} \to u$, where $u \in X$; contradiction. Relation (7.4) is proved.

Let $u^t \in X$ be the nearest element of v_t. We show that $\mathrm{dist}(u^t, u^{ct}) \to 0$, $t \to \infty$ (7.5) uniformly with respect to $c \in [1, 2]$. Assume that this is not so. Then $\mathrm{dist}(u^{t_m}, u^{c_m t_m}) \geq \varepsilon > 0$ (7.6) for some sequences $c_m \in [1, 2]$, $t_m \to \infty$. We have

$$u^{c_m t_m}(z) = v_{c_m t_m}(z) + o(1) = c_m^{-\rho} v_{t_m}(c_m z) + o(1) = c_m^{-\rho} u^{t_m}(c_m z) + o(1) = u^{t_m}(z) + o(1),$$

since $c^{-\rho}u(cz) = u(z)$ for any $u \in X$ and $c > 0$. We have obtained a contradiction with (7.6) and this proves (7.5).

If $u_t = u(.;\ \theta_0(t))$, then from (7.5) there follows that $\theta_0(t) - \theta_0(ct) \to 0$, $t \to \infty$ uniformly with respect to $c \in [1, 2]$. From (7.4) we obtain that $v_t(z) = u(z;\ \theta_0(t)) + o(1)$ in L^1_{loc} for $t \to \infty$. Finally, with the aid of V. S. Azarin's theorem on convergence with respect to the l-measure, we obtain (1.4).

The remaining assertions of Theorem 1 can be derived easily from (1.4), (1.5). Indeed, from the asymptotic formula (1.4), integrating along curves that differ little from rays and go around the exceptional set C_0, we obtain that for some $b_j \in \mathbb{C}$ we have

$$\log\frac{1}{|f(re^{i\theta}) - b_j|} = \pi r^\rho l_1(r)|\cos\rho(\theta - l_2(r))| + o(r^\rho l_1(r)),$$
$$\frac{\pi}{2\rho}(2j-3) \leq \theta - l_2(r) \leq \frac{\pi}{2\rho}(2j-1),$$

when $re^{i\theta} \notin C_0$, $r \to \infty$ uniformly with respect to θ. From here and from (1.5) we obtain at once properties 2), 3) from the formulation of Theorem 1.

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