ON THE FRONTIER OPERATOR IN BOOLEAN ALGEBRAS WITH A CLOSURE

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Kuratowski has shown in [1] that a topology on a set can be determined by a closure operator which is defined by the following axioms:

1°) $(A \cup B)^r = A^r \cup B^r$;

2°) $A \subset A^r$;

3°) $A^{rr} = A^r$;

4°) $0^r = 0$,

and also proved that by applying the closure and the complementation operators to an arbitrary set in any sequence we cannot obtain more than 14 different sets and investigated all possible inclusions between these sets (see also [2], p. 48).

Zarycki [3] has studied the frontier operator

$$A^f = A^r \cap A^{cr} \tag{1}$$

and has proved that we cannot get more than six different sets from any given set by means of the frontier and the complementation operators.

The problem naturally arises as to how many different sets can be obtained from any given set by applying all the three operators (closure, complementation, and frontier) in an arbitrary sequence.

Let an arbitrary Boolean algebra $\mathfrak{A}$ be given. Let us denote the zero and the unity of this Boolean algebra by 0 and 1, respectively, and the complement of an element $A \in \mathfrak{A}$ by A^c. Let r be an unary operator satisfying the system of the axioms 1°)-4°) and the operator f be defined by Eq. (1).

Let us prove the following theorem.

THEOREM. Not more than 34 different elements including 0 and 1 can be obtained by applying the operators f, c, and r to any element $A \in \mathfrak{A}$ in an arbitrary sequence.

Proof. Let us observe some relations:

$$A^{rcrcrcr} = A^{rcr} \quad \text{(see [1, 2])}; \tag{2}$$

$$A^{rfcr} = 1 \tag{3}$$

(indeed, $A^{rfcr} = (A^r \cap A^{rcr})^{cr} = A^{rcr} \cup A^{rcrcr} \supset A^{rcr} \cup A^{rcrc} = 1$);

$$A^{fr} = A^f; \tag{4}$$

$$A^{cf} = A^f \tag{5}$$

((4) and (5) follow immediately from (1));

$$A^{rcrf} = A^{rcrcrf} \tag{6}$$

(indeed, $A^{rcrcrf} = A^{rcrcr} \cap A^{rcrcrcr} = A^{rcrcr} \cap A^{rcr} = A^{rcrf}$);

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$$A^{rff} = A^{rf} \text{ (see [3]);} \tag{7}$$

$$A^{fff} = A^{ff} \text{ (see [2, p. 61]).} \tag{8}$$

Let us now proceed to prove the theorem. Applying each of the three operators to a given element A we get A^r, A^c, and A^f. If further only the operators c and r are applied, then we obtain in all 14 elements from A^r and A^c (see [1]):

$$A, A^r, A^{rc}, A^{rcr}, A^{rcrc}, A^{rcrcr}, A^{rcrcrc}, A^c, A^{cr}, A^{crc}, A^{crcr}, A^{crcrc}, A^{crcrcr}, A^{crcrcrc}. \tag{9}$$

Applying the operators r and c to A^f we obtain the six elements

$$A^f, A^{fc}, A^{fcr}, A^{fcrc}, A^{fcrcr}, A^{fcrcrc}, \tag{10}$$

since by virtue of (2) and (4) we have $A^{fcrcrcr} = A^{frcrcrcr} = A^{frcr} = A^{fcr}$.

Applying the operator f to the 20 elements obtained above, we obtain the six new elements

$$A^{rf}, A^{rcrf}, A^{crf}, A^{crcrf}, A^{ff}, A^{fcrf}. \tag{11}$$

Indeed, those of these 20 elements [see (9) and (10)] in which the last operator is c cannot be considered by virtue of (5). We shall not obtain any new elements other than those of (11) from the elements A^{rcrcr} and A^{crcrcr} by applying the operator f by virtue of (6), and $A^{fcrcrf} = A^{frcrcrf} = A^{frcrf} = A^{fcrf}$ by virtue of (4) and (6).

Obviously, by applying the operators r and c to the elements of (9) and (10) we do not get new elements [according to the definition of the groups of elements (9) and (10)]; the application of the operator r to the elements of (11) does not lead to new elements by virtue of (4). Applying c to (11), we get six more elements

$$A^{rfc}, A^{ffc}, A^{rcrfc}, A^{crfc}, A^{crcrfc}, A^{fcrfc}. \tag{12}$$

If now the operator f is applied, we shall obtain either the elements of (11) by virtue of (7) and (8), or the elements of (12) by virtue of (5).

Applying the operator r once more to an arbitrary element of (12), by virtue of (3) we obtain the only new element 1. It now remains only to apply the operator c or f to 1 and obtain the last new element 0. Thus, we obtain 34 different elements.

Let us now obtain the inclusions holding good between these elements in general.

The inclusions $A^{rc} \subset A^c \subset A^{cr}$, $A^{rcrcrc} \subset A^{rcr} \subset A^{crcrcr} \subset A^{cr}$, $A^{rc} \subset A^{rcrcrc} \subset A^{crcrc} \subset A^{crcrcr}$, $A^{crcrcrc} \subset A^{crcr} \subset A^{rcrcr} \subset A^r$, $A^{crc} \subset A^{crcrcrc} \subset A^{rcrc} \subset A^{rcrcr}$, $A^{crc} \subset A \subset A^r$ have been proved in [1] and the inclusions $A^{ff} \subset A^f$, $A^{fc} \subset A^{ffc}$ have been proved in [3]. We shall use these for obtaining the remaining inclusions.

The following inclusions follow from (1):

$$A^f \subset A^r,\ A^f \subset A^{cr},\ A^{rf} \subset A^{rcr},\ A^{rcrf} \subset A^{rcrcr},\ A^{ff} \subset A^{fcr},\ A^{crf} \subset A^{crcr},\ A^{crcrf} \subset A^{crcrcr},\ A^{fcrf} \subset A^{fcrcr}.$$

From $A^f \subset A^{cr}$, $A^f \subset A^r$, the axiom 2° and $A \subset B \Rightarrow B^c \subset A^c$, we have

$$A^{fcrc} \subset A^{fcrcr},\ A^{rcr} \subset A^{fcr},\ A^{fcrcr} \subset A^{rcrcr},\ A^{fcrcr} \subset A^{crcrcr}.$$

Moreover it follows from $A^{fcr} = (A^r \cap A^{cr})^{cr} = A^{rcr} \cup A^{crcr}$ that $A^{crcr} \subset A^{fcr}$. On applying $A^{rcrcr} \subset A^r$ to A^f, by virtue of (4) we get $A^{fcrcr} \subset A^f$.

Further, $A^{fcrf} \subset A^{ff}$ follows from $A^{fcrf} = A^{fcr} \cap A^{fcrcr}$, $A^{ff} = A^f \cap A^{fcr}$, and $A^{fcrcr} \subset A^f$.

The inclusions $A^{rcrf} \subset A^{rf}$ and $A^{crcrf} \subset A^{crf}$ hold good since

$$A^{rf} = A^r \cap A^{rcr} \supset A^r \cap A^{rcr} \cap A^{rcrcr} = A^{rcr} \cap A^{rcrcr} = A^{rcrf}.$$

The inclusions $A^{rf} \subset A^{ff}$ and $A^{crf} \subset A^{ff}$ follow from

$$A^{rf} = A^r \cap A^{rcr} \subset A^r \cap A^{rcr} \cap A^{cr} \subset A^r \cap A^{cr}$$
$$\cap (A^{rcr} \cup A^{crcr}) = A^f \cap A^{fcr} = A^{ff}.$$

Moreover, there are inclusions between the above-mentioned 34 elements which follow from the law $A \subset B \Rightarrow B^c \subset A^c$ and transitivity.

The following example shows that, in general, all the 34 elements obtained above are different and in the general case only the inclusions obtained above by us hold good between them.

Let us take the set of all subsets of the real line with the operations of union and intersection and the complementation and the closure operators defined in the natural manner as the Boolean algebra $\mathfrak{A}$. As A let us take the set

$$A = (0, 1) \cup (1, 2] \cup \{Q \cap (2, 3)\} \cup \{4\} \cup [5, 6],$$

where Q is the set of all rational numbers. A direct verification shows that the set A possesses all the required properties.

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