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ESTIMATING THE CHARACTERISTIC EXPONENTS OF POLYNOMIALS

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1. We consider a polynomial f of degree $d \geq 2$, and we denote its n-th iteration by f^n. The results of the theory of iterations that are used in the present article may be found in [1, 2].

A root of the equation $f^n z = z$ is called a periodic point (with period n). The quantity $\chi(z) = \frac{1}{n}\log|(f^n)'(z)|$ is the characteristic exponent of this point. When the Julia set of the polynomial f is connected, we have

$$\chi(z) \leq 2\log d, \tag{1.1}$$

for any periodic point z, and this bound is sharp only when the Julia set is a line segment with z for its end [3]. In the present paper we obtain an upper bound for $\chi(z)$ for arbitrary polynomials, as well as a lower bound for $\chi(z)$ for the case in which the Julia set is totally disconnected.

We set

$$u_f(z) = \lim_{n\to\infty} \frac{1}{d^n} log^+|f^n(z)|. \tag{1.2}$$

This limit exists and is a subharmonic function in C ([4] is a standard reference for the theory of subharmonic functions). The function u_f is nonnegative and continuous on C. It is harmonic and positive in the domain $D = \{z: f^n z \to \infty,\ n \to \infty\}$, and $u_f(z) = 0$ in $C \setminus D = K$. We have the functional equation

$$u_f \circ f = du_f. \tag{1.3}$$

The Riesz measure μ_f of the function u_f is concentrated in the Julia set $J = \partial D = \partial K$. This is the only probability measure in C that has the following property: For any Borel set $E \subset C$ on which the function f is univalent, we have

$$d\mu_f(E) = \mu_f(fE). \tag{1.4}$$

The measure μ_f is called the equilibrium measure or the measure of maximum entropy.

Let $c_1, c_2, \ldots, c_{d-1}$ be all of the critical points (with zero derivative) of the polynomial f. We set

$$a = \max\{u(c_j): 1 \leq j \leq d-1\}, \tag{1.5}$$

$$b = \min\{u(c_j): 1 \leq j \leq d-1\}. \tag{1.6}$$

The numbers a and b are natural parameters characterizing the degree of disconnection of the Julia set: $a = 0$ if and only if J is connected; on the other hand, J is a Cantor set (totally disconnected) if $b > 0$. We should also note the connection between the number a and mean of the characteristic exponent

$$\chi_f = \int \log|f'|\, d\mu_f.$$

We have

$$\chi_f = \log d + \sum_{j=1}^{d-1} u_j(c_j),$$

so $a \leq \chi_f - \log d \leq (d-1)a$. In particular, $\chi_f = \log d$ if and only if $a = 0$, i.e., J is connected.

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Theorem 1.1. *If $f(z) = z^d + c$, $c \in C$, then*

$$\chi(z) \leq (d-1)a + 2\log d \leq 2\chi_f \tag{1.7}$$

for any periodic point z.

Let u be a subharmonic function, let μ be its Riesz measure, and let z_0 be some point at which $u(z_0) = 0$. We set

$$n(r, u, z_0) = \mu(\{z: |z - z_0| \leq r\}), \quad N(r, u, v_0) = \int_0^r n(t, u, u_o)\,\frac{dt}{t}.$$

Because $u(z_0) = 0$, the Jensen formula yields

$$N(r, u, z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta.$$

We define the order of the measure μ at the point z_0 as follows:

$$\rho = \varliminf_{r\to 0}\frac{\log N(r, u, z_0)}{\log r} = \varliminf_{r\to 0}\frac{\log n(r, u, z_0)}{\log r}.$$

It is easy to see that the order of the measure μ is the same as the quantity

$$\rho(u, z_0) = \varliminf_{r\to 0}(\log\max_\theta u(z_0 + re^{i\theta}))/\log r,$$

so it can also be called the order of the function u at the point z_0.

Theorem 1.2. *For any polynomial f and any point $z_0 \in J(f)$ we have*

$$\rho(u_f, z_0) \geq \frac{1}{\pi}\operatorname{arcctg}\frac{ad}{\pi},$$

where the number a is given by formula (1.5).

Corollary 1.3. *For any periodic point we have*

$$\chi(z) \leq \frac{\pi\log d}{\operatorname{arcctg}\frac{ad}{\pi}}. \tag{1.8}$$

If $a = 0$, then (1.7) and (1.8) become precisely bound (1.1). For small a we have

$$\frac{\pi\log d}{\operatorname{arcctg}\frac{ad}{\pi}} = 2\log d + \frac{4ad\log d}{\pi^2} + o(a), \quad a \to 0.$$

Thus, for $d \leq 8$ and small a inequality (1.8) provides a stronger result than Theorem 1.1.

For an arbitrary point $z_0 \in J$ we define the (upper) characterstic exponent according to the formula

$$\chi(z_0) = \varlimsup_{n\to\infty}\frac{1}{n}\log|(f^n)'(z_0)|.$$

Recall that a polynomial is said to be hyperbolic if the trajectories of all of its critical points are attracted to attracting cycles.

Corollary 1.4. *If f is a hyperbolic polynomial, then (1.8) is satisfied for any point $z \in J(f)$.*

The proof of Theorem 1.2 uses the following result from the theory of subharmonic functions; this result is also of independent interest. We set

$$A(r, u, z_0) = \inf_\theta u(z_0 + re^{i\theta}).$$

Theorem 1.5. *Let u be a subharmonic function in the neighborhood of a point z_0, $u(z_0) = 0$, and assume that the order of the function u at the point z_0 is ρ. Then*

$$\varlimsup_{n\to 0}\frac{A(r, u, z_0)}{n(r, u, z_0)} \geq \pi\operatorname{ctg}\pi\rho.$$

This theorem is overshadowed by the so-called $\cos\pi\rho$ inequalities of the theory of entire and subharmonic functions (see, for example, [5-7]). Its proof is a modification of arguments of [6].

We now consider the case of a totally disconnected Julia set in which it is possible to obtain a uniform lower bound for the characteristic exponent.

Theorem 1.6. *Let a and b be given by formulas (1.5) and (1.6), and assume that k is determined from the conditions $a < d^k b \le da$. Then*

$$\chi(z) \ge \frac{a + d^k(d-2)b}{(d-1)^k} \ge (d-1)b \tag{1.9}$$

for any periodic point z.

Corollary 1.7. *The Hausdorff dimension of the Julia set satisfies the inequality*

$$\mathrm{HD}(J) \le \frac{(d-1)^k \log d}{a + d^k(d-2)b} \le \frac{\log d}{(d-1)b}.$$

In §2 we will prove the following asymptotic expressions for $c \to \infty$ for the family of functions $f_c(z) = z^d + c$, $c \in \boldsymbol{C}$:

$$a_c = b_c = \frac{1}{d}\log|c| + o(1), \tag{1.10}$$

$$\chi(z) = \frac{d-1}{d}\log|c| + o(1), \tag{1.11}$$

for any periodic point $z = z_0$. It then follows immediately that bounds (1.7) and (1.9) are asymptotically sharp when $c \to \infty$, while (1.8) in the case under consideration differs from sharp by the factor $\log d$.

2. Proof of Theorem 1.1. We first prove (1.10) and (1.11). The polynomial f_c has a unique critical point 0 of multiplicity $d-1$. Thus,

$$a_c = b_c = \lim_{n\to\infty} \frac{1}{d^n}\log^+|f_c^n(0)| = \frac{1}{d}\log|c| + \sum_{k=1}^{\infty} \frac{1}{d^{k+1}}\log^+\left|1 + \frac{c}{(f_c^k(0))^d}\right|.$$

When we let c go to ∞ we obtain (1.10).

If $z_1 = z_c$ is a periodic point with period n, then

$$z_i = z_{i-1}^d + c, \quad i = 2, \dots, n; \quad z_1 = z_n^d + c. \tag{2.1}$$

It follows that $z_i \to \infty$ as $c \to \infty$. As a result,

$$\prod_{i=1}^n \left(1 + \frac{c}{z_i^d}\right) = \frac{1}{(z_1 \cdot \dots \cdot z_n)^{d-1}} \to 0, \quad c \to \infty,$$

so the modulus of at least one factor $(1+c/z_j^d)$, $j = j(c)$, is small. We now find from (2.1) that $z_i/z_{i-1} \to 1$, $z_n/z_1 \to 1$, so $|z_i|^d \sim |c|$, $c \to \infty$, $1 \le i \le n$. As a result,

$$\left(\prod_{i=1}^n |f_c(z_i)|\right)^{1/n} \sim |c|^{(d-1)/d}, \quad c \to \infty,$$

which proves (1.11).

We can now complete the proof of Theorem 1.

We have

$$a_c = \lim_{n=\infty} \frac{1}{d^n}\log^+|f_c^n(0)|.$$

The function $c \mapsto a_c$ is continuous [8] and subharmonic in $\boldsymbol{C}$. It is equal to zero on the set $M = \{c\colon J(f_c) \text{ is connected}\}$. The complement $U = \bar{\boldsymbol{C}} \setminus M$ is connected (by the principle of the maximum), and the function a_c is positive and harmonic in U. Thus, da_c is a Green's function for the domain U with a pole at ∞.

Fix a natural number n. It is easy to see that

$$\chi_n(c) = \max\{\chi(z)\colon f_c^n z = z\}$$

is a subharmonic function in $\boldsymbol{C}$. In virtue of (1.1), we have $\chi_n(c) \le 2\log d$ on the set M, and in the neighborhood of ∞, by (1.10) and (1.11), we have $\chi_n(c) \le (d-1)a_c + o(1)$.

Application of the principle of harmonic majorants to the domain U yields $\chi_n(c) \leq 2\log d + (d-1)a_c$, i.e., (1.7).

3. Proof of Theorem 1.5. Without loss of generality, we assume that $z_0 = 0$. Furthermore, we assume that u is a subharmonic function in C and we have

$$u(z) = O(\log|z|), \quad z \to \infty \tag{3.1}$$

(any function that is subharmonic on a compactum of C can be continued in C with property (3.1)). The function (1.2), to which we are preparing to apply Theorem 1.5, already has property (3.1). We obtain the following representation from (3.1) and $u(0) = 0$:

$$u(z) = \int \log\left|1 - \frac{z}{\zeta}\right| d\mu_\zeta. \tag{3.2}$$

We set $A(r,u) = A(r,u,0)$, $n(r,u) = n(r,u,0)$, $N(r,u) = N(r,u,0)$. Let

$$v(z) = \int \log\left|1 - \frac{z}{\zeta}\right| d\nu_\zeta,$$

where ν is a measure that is concentrated on a negative ray and has the computational function $n(r,v) \equiv n(r,u)$. It follows from the inequality $\log|1-|u|| \leq \log|1-u| \leq \log(1+|u|)$, $u \in C$, that $A(r,u) \geq A(r,v) = v(-r)$. It is therefore sufficient to prove the theorem for the function v instead of u. We will need the following

Lemma 3.1 (on Polya peaks). *Let Φ be an increasing function, $\Phi(0) = 0$,*

$$\rho = \varliminf_{r\to 0} \frac{\log\Phi(r)}{\log r} < \infty.$$

Then there exist sequences $r_k \to 0$, $\varepsilon_k \to 0$, such that

$$\Phi(r) \leq \Phi(r_k)\left(\frac{r}{r_k}\right)^\rho (1+\varepsilon_k), \quad \varepsilon_k r_k \leq r \leq \varepsilon_k^{-1} r_k. \tag{3.3}$$

If we substitute $r \to \infty$, $r_k \to \infty$ for $r \to 0$, $r_k \to 0$ in (3.3) and reverse the inequality, we obtain a well-known proposition that is frequently used in number theory and the theory of meromorphic functions of finite order (for the proof, see, for example, [9]). Our formulation can be reduced to the standard statement by using the substitution $\Psi(r) = 1/\Phi(1/r)$. Setting $\Phi(r) = N(r,v)$ in the lemma, we obtain a sequence of Polya peaks $r_k \to 0$ such that

$$N(r,v) \leq N(r_k,v)\left(\frac{r}{r_k}\right)^\rho (1+\varepsilon_k), \quad \varepsilon_k r_k \leq r \leq \varepsilon_k^{-1} r_k. \tag{3.4}$$

We now consecutively examine the subharmonic functions

$$v_k(z) = \frac{v(r_k z)}{N(r_k,v)}. \tag{3.5}$$

It is obvious that

$$v_k(z) = \int \log\left|1 - \frac{z}{\zeta}\right| d\nu_k, \tag{3.6}$$

where the measures ν_k are defined thus:

$$\nu_k(E) = \frac{\nu(r_k E)}{N(r_k,v)}.$$

We have the relations

$$n(r,v_k) = \frac{n(rr_k,v)}{N(r_k,v)}, \tag{3.7}$$

$$N(r,v_k) = \frac{N(rr_k,v)}{N(r_k,v)}. \tag{3.8}$$

It now follows from the bound $n(r,v_k) \leq N(re,v_k)$ and (3.4) that the measures ν_k are uniformly bounded on compacta. Choosing a subsequence, we assume that $\nu_k \to \nu_0$ (convergence in the space conjugate to the space of continuous finite functions).

Then $v_k \to w_0$, where $w_0(z) = \int \log\left|1 - \frac{z}{\zeta}\right| d\nu_0$. The convergence $v_k \to w_0$ occurs in mean with respect to area in each compactum in $\boldsymbol{C}$, and also in mean with respect to the 1-measure on each compactum of $\boldsymbol{R}$. We have

$$n(r, v_k) \to n(r, w_0), \quad N(r, v_k) \to N(r, w_0). \tag{3.9}$$

It follows from (3.4) and (3.8) that

$$N(r, w_0) \leq r^\rho, \quad 0 \leq r < \infty, \quad N(1, w_0) = 1. \tag{3.10}$$

The theorem will be proved if we prove that

$$\frac{w_0(-1)}{n(1, w_0)} \geq \pi \operatorname{ctg} \pi\rho. \tag{3.11}$$

For this we consider the auxilliary function

$$w_1(re^{i\theta}) = \rho^2 \int_0^\infty \log\left|1 + \frac{re^{i\theta}}{t}\right| t^{\rho-1}\, dt = \frac{\pi\rho r^\rho}{\sin \pi\rho} \cos \rho\theta, \quad |\theta| \leq \pi$$

(the equation follows from Jensen's formula). We have

$$n(r, w_1) = \rho r^\rho, \tag{3.12}$$

$$N(r, w_1) = r^\rho, \tag{3.13}$$

$$w_1(-1) = \pi\rho \operatorname{ctg} \pi\rho. \tag{3.14}$$

Note that the function $N(r, w_0)$ is convex with respect to logarithms. As a result, it follows from (3.10) that $N(r, w_0)$ is differentiable at the point 1 and

$$n(1, w_0) = \frac{d}{d \log r} N(r, w_0)\bigg|_{r=1} = \frac{d(r^\rho)}{d \log r}\bigg|_{r=1} = \rho. \tag{3.15}$$

We will now show that $w_0(-1) \geq w_1(-1)$. Both of the functions w_0 and w_1 are harmonic in the plane cut along a negative ray. We set

$$w_j^*(re^{i\theta}) = \frac{1}{2\pi} \int_{-\theta}^{\theta} w_j(re^{i\theta})\, d\varphi, \quad 0 \leq \theta \leq \pi, \quad j = 0, 1.$$

It is easy to see that the functions w_j^* are harmonic in the upper halfplane (w_j^* is a trivial special case of Baernstein's *-function; see, for example, [10, 11]).

We have $w_0^*(r) = w_1^*(r) = 0$, $r > 0$,

$$w_0^*(-r) = N(r, w_0) \leq r^\rho = N(r, w_1) = w_1^*(-r), \quad r > 0, \tag{3.16}$$

in virtue of (3.10) and (3.13). It follows, by the Phragmen–Lindelof theorem, that

$$w_0^*(z) \leq w_1^*(z), \quad \operatorname{Im} z > 0. \tag{3.17}$$

Furthermore, in virtue of (3.10) and (3.13), we have $w_0^*(-1) = w_1^*(-1)$, which, together with (3.17), yields

$$\frac{\partial w_0^*(e^{i\theta})}{\partial \theta}\bigg|_{\theta=\pi} \geq \frac{\partial w_1^*(e^{i\theta})}{\partial \theta}\bigg|_{\theta=\pi}.$$

But $\frac{\partial w_j^*(e^{i\theta})}{\partial \theta}\Big|_{\theta=\pi} = \frac{1}{\pi} w_j(-1)$, so $w_0(-1) \geq w_1(-1) = \pi\rho \operatorname{ctg} \pi\rho$, which, together with (3.15), yields (3.11).

The theorem is proved.

4. Proof of Theorem 1.2 and Corollaries 1.3 and 1.4. To prove Theorem 1.2 we consider two cases:

a) the point z_0 is contained in a connected Julia-set component with more than one point. Then $A(r, u, z_0) \equiv 0$ for a subharmonic function u and sufficiently small r; by Theorem 1.5, we now have $\rho \geq \frac{1}{2} \geq \frac{1}{\pi} \operatorname{arcctg} \frac{ad}{\pi}$;

b) the point z_0 is a comnnected component of the Julia set J. We set $E_0 = \{z\colon u(z) \le a\}$. The set E_0 is connected, so a is the largest critical value of the function u. Let E_k be the connected component of the set $f^{-k}(E_0)$ containing the point z_0. In other words, E_k is the connected component of the set $\{z\colon u(z) \le ad^{-k}\}$ containing the point z_0 (see (1.3)). It follows from (1.4) that

$$\mu(E_k) \ge d^{-k} \tag{4.1}$$

($\mu(E_0) = 1$, since $\operatorname{supp}\mu = J \subset E_0$). Since z_0 is the connected component of the set J, we have

$$\bigcap_{k=0}^{\infty} E_k = \{z_0\}. \tag{4.2}$$

Let D_r be a circle with center at the point z_0 and radius r small enough for $C_r = \partial D_r$ to intersect E_0. Let $k(r)$ be the smallest natural number such that $E_{k(r)} \subset D_r$ (the existence of such a number is implied by (4.2)). It follows from (4.1) that

$$\mu(D_r) \ge \mu(E_{k(r)}) \ge d^{-k}. \tag{4.3}$$

By the definition of the number $k(r)$, the set $E_{k(r)-1}$ is not contained in D_r. Since $E_{k(r)-1}$ is connected and contains z_0, it must intersect C_r, so

$$A(r, u, z_0) \le ad^{-k+1}. \tag{4.4}$$

It follows from (4.3) that $n(r, u, z_0) \ge d^{-k}$. Thus, for all sufficiently small $r > 0$,

$$\frac{A(r, u, z_0)}{n(r, u, z_0)} \le ad.$$

Application of Theorem 1.5 finishes the proof of Theorem 1.2.

To prove the corollaries we will need

Proposition 4.1. *Let $z_0 \in J(f)$, and use $r_n(z_0)$ to denote the radius of the largest disk centered at $f^n z_0$ that contains a univalent branch g_n of the function f^{-n} with the property $g_n(f^n z_0) = z_0$. We assume that*

$$r(z_0) = \varliminf_{n\to\infty} r_n(z_0) > 0. \tag{4.5}$$

Then

$$\rho(u, z_0) = \frac{\log d}{\chi(z_0)}$$

(the upper characteristic exponent $\chi(z_0)$ for any point $z_0 \in J(f)$ is defined in §1).

Proof. The function g_n is univalent in the disk $\{z : |z - f^n z_0| < r(z_0)\}$. According to the "distortion theorem," half the disk $\{z : |z - f^n z_0| < \frac{1}{2} r(z_0)\}$ can be mapped onto an oval with bounded distortion E_n, where $t_n = \operatorname{diam} E_n \asymp |(f^n)' z_0|^{-1}$ (the symbol $\asymp$ indicates that a variable is bounded above and below by positive absolute constants). By the definition of the characteristic exponent,

$$\varliminf_{n\to\infty} \frac{\log t_n}{n} = -\chi(z_0).$$

On the other hand,

$$\mu(E_n) \asymp d^{-n},$$

$$\frac{\log \mu(E_n)}{n} \to -\log d.$$

It follows that

$$\varlimsup_{n\to\infty} \frac{\log t_n}{\log \mu(E_n)} = \frac{\chi(z_0)}{\log d}.$$

Because the function $n(t, u, z_0)$ is monotonic and $\log \mu(E_{n+1}) - \log \mu(E_n) = O(1)$, it follows that

$$\varlimsup_{r\to 0} \frac{\log r}{\log n(r, u, z_0)} = \frac{\chi(z_0)}{\log d}$$

or

$$\rho(u, z_0) = \varlimsup_{r\to 0} \frac{\log n(r, u, z_0)}{\log r} = \frac{\log d}{\chi(z_0)}.$$

We should note that condition (4.5) is satisfied in two cases:

a) f is an arbitrary polynomial and z_0 is a periodic point;

b) f is a hyperbolic polynomial and $z_0 \in J(f)$ is any point.

Thus, Corollaries 1.3 and 1.4 follow from Theorem 1.2 and Proposition 4.1.

5. Proof of Theorem 1.6 and Corollary 1.7.

We will use the method of extremal lengths [12], and we denote the modulus of a family of curves Γ by $M(\Gamma) = \lambda(\Gamma)^{-1}$, where λ is the extremal length. An immediate consequence of the definition of modulus is

Lemma 5.1. *Let Γ be a family of pairwise disjoint curves filling a domain U, and let $g\colon U \to g(U)$ be a holomorphic mapping with two properties:*

(i) if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, then $g(\gamma_1) \cap g(\gamma_2) = \emptyset$;

(ii) $g\colon \gamma \to g(\gamma)$ is a covering of degree no greater than N.

Then $M(\Gamma) \geq M(g(\Gamma))/N$.

We label the critical points $c_1, c_2, \ldots, c_{d-1}$ of the polynomial f so that $u_1 \geq u_2 \geq \cdots \geq u_{d-1}$, where $u_i = u_f(c_i)$, and, in particular, $u_1 = a$, $u_{d-1} = b$.

We can assume that $d^l u_i \neq u_j$, $i \neq j$, $l \in \boldsymbol{Z}$. If we can prove the theorem for this case, we can obtain the general case from the continuity of the mappings $(z, f) \mapsto u_f(z)$ and $(z, f) \mapsto \chi(z)$.

It follows from this assumption that each component of the level curve $L(\rho) = \{z\colon u(z) = \rho\}$ is either a simple closed real analytic curve, or a figure-eight shaped curve (which occurs when $\rho = u_l d^{-l}$, $l \in \boldsymbol{Z}_+$).

The Batcher function [2, 8] conformally maps the annulus $\{z\colon u_1 < u(z) < du_1\}$ onto the annulus $\{z\colon e^{u_1} < |z| < e^{du_1}\}$, so

$$M(\Gamma_0) = \frac{(d-1)u_1}{2\pi}, \tag{5.1}$$

where $\Gamma_0 = \{L(\rho)\colon u_1 < \rho < du_1\}$.

Let $z \in J$ and $n \in \boldsymbol{N}$. We use $\Gamma_n = \Gamma_n(z)$ to denote the set of components of the level lines $L(\rho)$, $u_1/d^n < \rho < u_1/d^{n-1}$, that include the point z. We now find a lower bound for $M(\Gamma_n(z))$. Note that for any singly-connected domain V bounded by a component of the level line $L(\rho)$, the mapping $f\colon V \to f(V)$ is an N-sheeted branching covering, where $N-1$ is equal to the number of critical points of the function f in V. As a result, the $M(\Gamma_n)$ are equal when $n \geq k$, where $k \in \boldsymbol{N}$ is given by the condition

$$u_1 d^{-k} < u_{d-1} < u_1 d^{-k+1}$$

(i.e., k is consistent with the conditions of Theorem 1.6).

The family Γ_k splits into two parts: the curves $\gamma_\rho \subset L_\rho$ that include the critical point c_{d-1} (if $u_{d-1} < \rho < u_1 d^{-k+1}$), and the curves that do not include critical points (if $u_1 d^{-k} < \rho < u_{d-1}$). The function f^k maps Γ_k onto Γ_0. It follows from (5.1), Lemma 5.1, and the properties of extremal lines that

$$M(\Gamma_k) \geq \frac{1}{2}\left(\frac{d^k u_{d-1} - u_1}{(d-1)^{k-1}} + \frac{du_1 - d^k u_{d-1}}{(d-1)^k}\right) \tag{5.2}$$

Now, let z_0 be a periodic point with period m, $\lambda = (f^m)' \times (z_0)$, $\chi(z_0) = \frac{1}{m}\log|\lambda|$,

$$\Gamma = \bigcup_{i=0}^{m-1} \Gamma_{n+1}(z_0). \tag{5.3}$$

The curves of the family Γ separate boundary components of the annulus K, which is bounded by certain curves $\gamma_1 \subset L(ad^{-n-m})$ and $\gamma_2 \subset L(ad^{-n})$. Since the family Γ_i is pairwise disjoint, we have, in view of (5.2),

$$M(K) \geq M(\Gamma) \geq \frac{m}{2\pi}\frac{a + d^k(d-2)b}{(d-1)^k} \tag{5.4}$$

($M(K)$ is the modulus of the annulus K [12]).

We now obtain an upper bound for $M(K)$. If n is large, the annulus K lies in a small neighborhood of the point z_0. By Schroder's theorem [2], there exists a holomorphic change of coordinates in the neighborhood of the point z_0, $\zeta = \psi(z), \psi(z_0) = 0$, that linearizes the transformation f^m. We set $K^* = \psi(K)$. The mapping $\zeta \mapsto \lambda^{-1}\zeta$ transforms the outer boundary component of the annulus K^* into the inner. If we choose a conformal metric with density $\rho(\zeta) = (2\pi|\zeta|)^{-1}$ in the annulus, we find that the length of closed curves separating the boundary components is no less than 1, and the area of the annulus in this metric is no more than $(2\pi)^{-1}\log|\lambda|$. As a result, the extremal length is larger than or equal to $2\pi(\log|\lambda|)^{-1}$ and

$$M(K) = M(K^*) \leq \frac{1}{2\pi}\log|\lambda|.$$

Together with (5.4), this relation proves Theorem 1.6.

In order to derive Corollary 1.7, we note that the polynomial f is hyperbolic if $b > 0$. In order to compute the Hausdorff dimension of the Julia set of a hyperbolic polynmial, we can use a thermodynamic formalism [13, 14].

We set

$$P(t) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{z\in\mathrm{Per}_n}|(f^n)'(z)|^{-t}, \quad t \in \boldsymbol{R}, \tag{5.5}$$

where Per_n is the set of points with period n. The limit in (5.5) exists and is called pressure. The function $t \mapsto P(t)$ is a strictly decreasing function and has a unique zero at the point $t = \mathrm{HD}(J) \geq 0$. It follows from Theorem 1.7 that $|(f^n)'(z)| \geq e^{n\chi}$ for any point $z \in \mathrm{Per}_n$, where χ satsifies (1.9). It thus follows that $P(t) \leq \log d - t\chi$, which implies the desired bound for $\mathrm{HD}(J)$.

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