ITERATIONS OF RATIONAL FUNCTIONS AND THE DISTRIBUTION OF THE VALUES OF THE POINCARÉ FUNCTIONS

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We consider a rational function R of degree $d \geq 2$. We assume that the function R has a repellent fixed point ζ, i. e. $R(\zeta) = \zeta$, $|R'(\zeta)| > 1$. According to Poincaré's theorem [1, Chap. VII], there exists a unique function f, meromorphic in C and satisfying the equation

$$f(\lambda z) = R[f(z)],\ \lambda = R'(\zeta), \tag{1}$$

and the initial conditions $f(0) = \zeta$, $f'(0) = 1$. Equation (1) is called the Poincaré equation and its solution f is called the Poincaré function. It is convenient to write Poincaré's equation in the form of the commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda\cdot} & \mathbb{C} \\ f \downarrow & & \downarrow f \\ \overline{\mathbb{C}} & \xrightarrow{R} & \overline{\mathbb{C}} \end{array} \tag{2}$$

Let R^n be the nth iterate of the function R. In the classical works of Julia, Fatou, and Lattés one has pointed out the close connection between the distribution of the values of the function f and the distribution of the roots of the equation $R^n(z) = a$. We shall continue the investigation of this connection, making use of the Nevanlinna theory of the distribution of the values of meromorphic functions [2, 3]. In particular, we shall give a new proof of the uniqueness of an invariant balanced measure of the function R and of the asymptotically uniform distribution of the roots of the equation $R^n(z) = a$ with respect to this measure [4, 5]. The definition of a balanced measure and the precise formulation of the result are given in Sec. 5.

All the facts regarding iterates of rational functions, used in this paper, can be found in [6, Chap. VIII].

1. Exceptional Values. By definition, the set E(R) of the exceptional values of a rational function R consists of those $a \in \overline{\mathbb{C}}$, such that the equation $R^n(z) = a$, $n \in N$, have in totality a finite number of roots. In other words, the points $a \in E(R)$ have only a finite number of antecedents. As it is known, card $E(R) \leq 2$.

The rational function R and S are said to be conjugate if $R \circ \varphi = \varphi \circ S$ for some linear fractional function φ. If card $E(R) = 2$, then the function R is conjugate with $z \mapsto z^{\pm d}$. If card $E(R) = 1$, then R is conjugate with a polynomial of degree d.

We denote by $E_P(f) = \{a \in \overline{\mathbb{C}} : f(z) \neq a,\ z \in \mathbb{C}\}$ the set of Picard exceptional values of the function f. If f is the Poincaré function for R, then $E_P(f) = E(R)$. In particular, f is entire if and only if R is a polynomial.

We need one elementary lemma.

LEMMA 1. If the equation $R^3(z) = a$ has a root of order d^3, then $a \in E(R)$.

For the sake of completeness, we give the proof of this lemma.

Assume that the equation $R^3(z) = a$ has a root of order d^3. Then it has only one root. In this case the equation $R(z) = a$ has a unique root a_{-1} of order d and the equation $R(z) = a_{-1}$ has a unique root a_{-2} of order d. Since also the equation $R(z) = a_{-2}$ has a unique root d, we conclude that among the points a, a_{-1}, a_{-2} at least two are equal (since the total number of critical points of the function R, taking into account multiplicities, is equal to $2d - 2$). From here it follows that $a \in E(R)$.

2. The Nevanlinna Characteristics [2, 3]. For an arbitrary function f, meromorphic in C, we set

$$N(r,\ a,\ f) = \sum_{|b_j| < r} \log\frac{r}{|b_j|} + k\log r,$$

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where the summation extends over all nonzero roots b_j of the equation $f(z) = a$, taking into account multiplicities, while k is the order of the value a at the point $z = 0$ (if $f(0) \neq a$, then $k = 0$). Further,

$$m(r, f) = m(r, \infty, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta,$$

$$m(r, a, f) = m\left(r, \frac{1}{f-a}\right), \quad a \in \mathbb{C},$$

$$T(r, f) = m(r, f) + N(r, \infty, f).$$

The quantity $T(r, f)$ is called the Nevanlinna characteristic of the function f. By the order of the function f we mean the number

$$\rho = \limsup_{r\to\infty} \log T(r, f)/\log r.$$

Nevanlinna's first fundamental theorem asserts that

$$m(r, a, f) + N(r, a, f) = T(r, f) - \log|f(0) - a| + \varepsilon(a, r),$$

where $|\varepsilon(a, r)| \le \log^+|a| + \log 2$. If $f(0) = a$, $f'(0) \neq 0$, then $\log|f(0) - a|$ has to be replaced by $\log|f'(0)|$.

According to Nevanlinna's second fundamental theorem, for any mutually distinct values $a_1, \ldots, a_n \in \overline{\mathbb{C}}$ we have

$$\sum_{j=1}^{n} m(r, a_j, f) < 2T(r, f) + Q(r, f; a_1, \ldots, a_n).$$

Here Q is the remainder, small with respect to $T(r, f)$. For functions of finite order we have $Q(r, f) = O(\log r)$, $r \to \infty$.

3. Valiron Exceptional Values of the Poincaré Function. A value $a \in \overline{\mathbb{C}}$ is said to be exceptional in the Valiron sense for the function f if

$$\limsup_{r\to\infty} \frac{m(r, a, f)}{T(r, f)} > 0.$$

The set of such exceptional values is denoted by $E_V(f)$. Obviously, $E_P(f) \subset E_V(f)$. The set $E_V(f)$ has always zero logarithmic capacity, but may have the power of the continuum [2, 3].

It is known that for a rational function R of degree d and for any meromorphic function f we have $T(r, R \circ f) = dT(r, f) + O(1)$, $r \to \infty$ (see, for example, [3, Chap. 1]). From here and from (1) for the Poincaré function we obtain

$$T(|\lambda| r, f) = (d + o(1))T(r, f), \ r \to \infty, \tag{3}$$

and, in particular, f has finite order

$$\rho = \log d / \log|\lambda| \tag{4}$$

and normal type. This is Valiron's result [1, Chap. VII].

THEOREM 1. Assume that f and R are related by the Poincaré equation. Then $E_V(f) = E(R)$.

Proof. First we note that if S is a rational function and $b_1, \ldots, b_q$ are all the roots of the equation $S(z) = a$, where the root b_j has order k_j, then

$$m(r, a, S \circ f) < \sum_{j=1}^{q} k_j m(r, b_j, f) + O(1), \ r \to \infty.$$

In fact, it is sufficient to prove this relation for $a = \infty$, $b_1, \ldots, b_q \in \mathbb{C}$. Then

$$|S(w)| < K\left(\sum_{j=1}^{q} |w - b_j|^{-k_j} + 1\right),$$

and we have

$$m(r, \infty, S \circ f) < \sum_{j=1}^{q} m(r, \infty, (f - b_j)^{-k_j}) + O(1) =$$

$$= \sum_{j=1}^{q} k_j m(r, b_j, f) + O(1).$$

Now we apply Lemma 1. If $a \notin E(R)$, then the order of the roots of the equation $R^{3n}(z) = a$ does not exceed $(d^3 - 1)^n$. From (1) there follows that $f(\lambda^{3n}z) = R^{3n} \circ f(z)$, $n \in N$. Therefore,

$$m(r, a, f) \leq \sum_{b:R^{3n}(b)=a} (d^3 - 1)^n m\left(\frac{r}{|\lambda|^{3n}}, b, f\right),$$

where the sum is taken over all the distinct roots of the equation $R^{3n}(z) = a$. From here, by the second fundamental theorem and taking into account (3), we conclude that

$$m(r, a, f) \leq (2 + o(1))(d^3 - 1)^n T\left(\frac{r}{|\lambda|^{3n}}, f\right) =$$
$$= (2 + o(1))\left(\frac{d^3 - 1}{d^3}\right)^n T(r, f), \quad r \to \infty.$$

Selecting the number n arbitrarily large, we obtain $m(r, a, f) = o(T(r, f)$, $r \to \infty$.

We have proved that $E_V(f) \subset E_P(f)$. The converse inclusion is obvious. Theorem 1 is proved.

We mention that, by a similar method, N. Yanagihara [7] has proved earlier that the function f has no Nevanlinna deficiency values, different from E(R), i.e.

$$\liminf_{r \to \infty} m(r, a, f)/T(r, f) = 0$$

for all $a \in \bar{\mathbb{C}} \setminus E(R)$.

4. The Equidistribution of the Preimages of Measures. Let f be an arbitrary meromorphic function and let μ be a measure in $\bar{\mathbb{C}}$. We lift the measure μ with the aid of the function f, setting for an arbitrary bounded Borel set $X \subset C$

$$(f^*\mu)(X) = \int_{\bar{\mathbb{C}}} n(a, X)\, d\mu_a, \tag{5}$$

where $n(a, X)$ is the number of the roots of the equation $f(z) = a$, belonging to X (multiplicities included). The locally finite measure $f^*\mu$ will be called the preimage of the measure μ under the action of f. Obviously, the operator f^* is linear. For example, if $\mu = \delta_a$ is the unit atomary measure, concentrated at the point $a \in \bar{\mathbb{C}}$, then $f^*\delta_a(X) = n(a, X)$.

In the sequel we investigate in this section meromorphic functions of finite order ρ and normal type, i.e.,

$$T(r, f) = O(r^\rho), \ r \to \infty. \tag{6}$$

Let W be the conjugate space of the space of continuous finite function in C (i.e., W is the space of locally finite charges in C), provided with the topology of weak convergence. We denote by L^1_{loc} the space of locally summable functions in C with the topology of mean convergence on each compactum. The subharmonic functions are contained in L^1_{loc} and we consider the dense subspace $\delta SH \subset L^1_{loc}$, consisting of differences of subharmonic functions. The Laplace operator extends to a linear operator Δ: $\delta SH \to W$, possessing the following continuity property: if $u_t \to 0$, $u_t \in \delta SH$, $t \to \infty$, and the variations of the charges Δu_t are bounded on compacta, uniformly with respect to t, then $\Delta u_t \to 0$.

Following V. S. Azarin [8], for each $t \in C$ we define the linear operators L_t: $\delta SH \to \delta SH$, T_t: $W \to W$ by the formulas

$$L_t u(z) = |t|^{-\rho} u(tz), \ (T_t \nu)(X) = |t|^{-\rho} \nu(tX).$$

Then $T_t\Delta = \Delta L_t$ for all $t \in C$.

The measures $\mu_1, \mu_2 \in W$ are said to be ρ-equidistributed if the charge $\nu = \mu_1 - \mu_2$ satisfies the condition $T_t\nu \to 0$, $t \to \infty$.

Remark. If f is a meromorphic function of finite order ρ and normal type, then for any probability measure μ in $\bar{\mathbb{C}}$ we have $f^*\mu(D(0, r)) = O(r^\rho)$, $r \to \infty$ (7), where $D(a, t) = \{z: |z - a| < t\}$. We prove (7). We shall assume that the integrals

$$\int_{\bar{\mathbb{C}}} \log|f(0) - a|\, d\mu_a, \quad \int_{\bar{\mathbb{C}}} \log^+|a|\, d\mu_a$$

are finite; otherwise, we perform the transformation

$$f(z) \mapsto \frac{1}{f(z + \zeta) - w}$$

with suitable ζ, $w \in \mathbf{C}$. Now, making use of (5), (6) and Nevanlinna's first fundamental theorem, we obtain

$$f^*\mu(D(0, r)) = \int_{\overline{\mathbb{C}}} f^*\delta_a(D(0, r))\, d\mu_a \leq \int_{\overline{\mathbb{C}}} N(er, a, f)\, d\mu_a \leq T(er, f) + O(1) = O(r^\rho), \ r \to \infty.$$

From (6) and (7) there follows that the family of functions $\{L_t \log|f|\}_{|t|\geq 1} \subset \delta SH$ and the family of measures $\{T_t f^*\mu\}_{|t|\geq 1} \subset W$ are precompact in δSH and W, respectively.

THEOREM 2. Let f be a meromorphic function of order ρ and normal type and let μ_j be probability measures in $\overline{\mathbb{C}}$ such that $\mu_j(E_V(f)) = 0$, $j = 1, 2$. Then the measures $f^*\mu_1$ and $f^*\mu_2$ are ρ-equidistributed.

This theorem is a weakened variant of a result of one of the authors, communicated in [9], in which instead of $\mu_j(E_V(f)) = 0$, $j = 1, 2$, one requires only that $\mu_j(\{a\}) = 0$, $j = 1, 2$, for each point $a \in E_V(f)$.

For the proof of Theorem 2 we require the following.

LEMMA 2. For any meromorphic function f there exist constants r_0 and C such that $m(r, a, f) \leq T(r, f) + C$, $a \in \overline{\mathbb{C}}$, $r \geq r_0$.

Proof. For the sake of simplicity, we restrict ourselves to the case when $f'(0) \neq 0$. (In the sequel, Theorem 2 and Lemma 2 are used only in this case). We select numbers $r_0 > 0$ and $\delta > 0$, so small that the function f be univalent in $D(0, r_0)$ and we should have $D(f(0), \delta) \subset fD(0, r_0)$.

Let $G(z, \zeta, V)$ be the Green function of the domain V with pole at the point ζ, defined to be equal to zero in $\mathbf{C} / V$. In this case, if $f(0) \neq a$, then $N(r_0, a, f) = G(f^{-1}a, 0, D(0, r_0)) = G(a, f(0), fD(0, r_0))$; here $f^{-1}a$ is an a-point of the function f, nearest to the origin. From the monotonicity of $N(r)$ and the maximum principle for $r \geq r_0$ we obtain

$$N(r, a, f) \geq N(r_0, a, f) \geq G(a, f(0), D(f(0), \delta)) = \log^+ \frac{\delta}{|f(0)-a|} \geq \log^+ \frac{1}{|f(0)-a|} + \log\delta.$$

Now from Nevanlinna's first fundamental theorem there follows that

$$\begin{aligned} m(r, a, f) &\leq T(r, f) - N(r, a, f) + \log\frac{1}{|f(0)-a|} + \varepsilon(a, r) \leq \\ &\leq T(r, f) - \log^+\frac{1}{|f(0)-a|} + \log\frac{1}{|f(0)-a|} + \log\frac{1}{\delta} + \log^+|a| + \\ &+ \log 2 \leq T(r, f) + C_f, \ a \neq f(0). \end{aligned}$$

If, however, $a = f(0)$, then the required inequality follows directly from the first fundamental theorem. Lemma 2 is proved.

Proof of Theorem 2. There exists a point $a \in \overline{\mathbb{C}}$, at which the logarithmic potentials of both measures μ_j are finite and $a \notin E_V(f)$. It is sufficient to prove that each measure $f^*\mu_j$ is ρ-equidistributed with $f^*\delta_a$. Assuming that $a = \infty$ (this can be achieved by replacing f by $1/(f - a)$), we arrive at the following situation.

Prove that the measures $f^*\mu$ and $f^*\delta_a$ are ρ-equidistributed under the condition that

$$\int_{\mathbb{C}} \log^+|a|\, d\mu_a < \infty, \tag{8}$$

$$\infty \notin E_V(f), \tag{9}$$

$$\mu(E_V(f)) = 0. \tag{10}$$

Following Frostman's method [2, Chap. X], we consider the logarithmic potential

$$u(w) = \int_{\mathbb{C}} \log|w - a|\, d\mu_a$$

(by virtue of (8) the integral converges for almost all w). Then $U = u \circ f \in \delta SH$, $\Delta U = 2\pi(f^*\mu - f^*\delta_\infty)$ and, by virtue of the continuity of the Laplacian, it is sufficient to prove that

$$L_t U(z) = |t|^{-\rho} U(tz) \to 0, \ t \to \infty. \tag{11}$$

We fix an arbitrary number $r_1 < \infty$. For $0 < r \le r_1$, $t \to \infty$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} |L_t U(re^{i\theta})|\, d\theta \leqslant \frac{1}{2\pi |t|^\rho}\int_0^{2\pi}\int_{\mathbb{C}} |\log|f(tre^{i\theta}) - a||\, d\mu_a\, d\theta$$

$$\leqslant \frac{1}{2\pi|t|^\rho}\int_0^{2\pi}\int_{\mathbb{C}}\left\{\log^+|f(tre^{i\theta})| - a| + \log^+\frac{1}{|f(tre^{i\theta}) - a|}\right\} d\mu_a\, d\theta \leqslant$$

$$\leqslant \frac{1}{|t|^\rho}\int_{\mathbb{C}}\{m(|t|r, \infty, f) + \log 2 + \log^+|a| + m(|t|r, a, f)\}\, d\mu_a \leqslant$$

$$\leqslant Cr^\rho \int_{\mathbb{C}} \frac{m(s, \infty, f) + \log 2 + \log^+|a| + m(s, a, f)}{T(s, f)}\, d\mu_a,$$

where $s = |t|r$. By virtue of (8) and Lemma 2, the integrand has a μ-summable majorant. By Lebesgue's theorem, we can take the limit under the integral sign for $s \to \infty$, while, by virtue of (9) and (10), this limit is equal to zero. Thus,

$$\int_0^{2\pi} |L_t U(re^{i\theta})|\, d\theta \rightrightarrows 0 \quad t \to \infty,$$

uniformly with respect to r, $0 < r \le r_1$. From here we obtain (11). The theorem is proved.

5. The Balanced Measure and the Equidistribution of the Roots of the Equation $R^n z = a$. Let M be the set of all probability measures in $\bar{\mathbb{C}}$ with the property $\mu(E(R)) = 0$. We define an operator Q: M → M in the following manner:

$$Q\mu = \frac{1}{d} R^* \mu. \tag{12}$$

A measure $\mu \in M$ is said to be balanced if $Q\mu = \mu$. Roughly speaking, this means that for each Borel set $E \subset \bar{\mathbb{C}}$ the measure $\mu(E)$ is distributed equally among the preimages of the set E under the action of the function R.

THEOREM 3. For each rational function R there exists a unique balanced measure μ_R and, moreover, for each measure $\mu \in M$ we have $Q^n\mu \to \mu_R$, $n \to \infty$ (13).

If R is a polynomial, then the measure μ_R coincides with the balanced (in the sense of potential theory) measure of the Julia set J(R). In this case Theorem 3 has been proved by Brolin [10]. In the general case Theorem 3 has been proved by M. Yu. Lyubich [4] and, independently, by Freire, Lopes, and Mañé [5]. The proof in [4] is based on the investigation by means of functional analysis of an operator $A: C(\bar{\mathbb{C}}) \to C(\bar{\mathbb{C}})$, for which $Q = A^*$.

In the sequel we need the following sample.

LEMMA 3. Let R be a rational function, $\mu \in M$. Then for each neighborhood U of the set E(R) and for any number $\varepsilon > 0$ there exists an index N such that $(Q^n\mu)(U) < \varepsilon$ for $n \ge N$.

The proof of this lemma follows directly from the description of the set E(R), given in Sec. 1.

Proof of Theorem 3. The existence of the measure μ_R is established with the aid of the usual N. N. Bogolyubov—N. M. Krylov construction. Let M be the set of the probability measures ν on $\bar{\mathbb{C}}$ such that $\nu(E(R)) = 0$. We consider the sequence of Cesàro means

$$\nu^{(N)} = \frac{1}{N}\sum_{s=0}^{N-1} Q^s\delta_a, \quad a \notin E(R).$$

Clearly, $\nu^{(N)} \in M$. Let ν be some limit measure for the sequence $\nu^{(N)}$. Then ν is a probability measure and $Q\nu = \nu$. From Lemma 3 there follows that $\nu \in M$.

First we assume that R has a repellent fixed point ζ and we consider the Poincaré equation (1). We denote by T the operator T_λ, defined in Sec. 4, where $\lambda = R'(\zeta)$.

By virtue of the relation (4) we have $d = |\lambda|^\rho$ and, therefore, for each charge $\nu \in W$ and any bounded Borel set $X \subset C$ we have $T\nu(X) = d^{-1}\nu(\lambda X)$. From the definition of the operators Q, T, f^* and the diagram (2) there follows that $f^*Q = Tf^*$. Obviously, the operator f^*: M → W is continuous and injective.

For any measure $\mu \in M$, by virtue of Theorems 1 and 2 we have

$$0=\lim_{n\to\infty} T^n (f^*\mu_R - f^*\mu) = f^*\mu_R - \lim_{n\to\infty} T^n f^*\mu,$$

from where we obtain at once the uniqueness of the invariant measure μ_R and (13).

We get rid of the assumption that R has a repellent fixed point. We select $k \in N$ so that R^k should have repellent fixed points (this can be done since the number of all nonrepellent periodic points of the function R is finite).

Obviously, the measure μ_R, constructed by the Krylov—Bogolyubov method, is balanced also for R^k and, according to what has been proved, μ_R is the unique balanced measure for R.

For any measure $\mu \in M$ we have $Q^{kn}\mu \to \mu_R$, $n \to \infty$. Then for any $q \in \{0, 1, 2, \ldots, k-1\}$ we have $Q^{kn+q}\mu = Q^{kn}(Q^q\mu) \to \mu_R$, $n \to \infty$, and, therefore, (13) holds.

Theorem 3 is proved.

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