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INDEPENDENCE OF SOME POLYNOMIAL STATISTICS AND OF THE SAMPLE MEAN

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Let $x = (x_1,\dots,x_n)$ be a random vector in $\mathbf{R}^n$ with independent components. By a polynomial statistic we mean a random variable $P(x) = P(x_1,\dots,x_n)$, where P is a polynomial in the coordinates of the vector x. We assume that x is a sample with replacement, i.e., that the random variables have the same distribution F(t). One of the important characterization problems of mathematical statistics consists of determining the functions F(t) for which two polynomial statistics, $P_1(x)$ and $P_2(x)$, can be independent random variables. When one of the statistics is linear, the general method of solving such problems is that of differential equations. By this method the case when P_1 is a linear form and P_2 a quadratic one has been relatively completely treated ([1], Sec. 4.2).

In the present note we study third-degree statistics independent of a linear form. We shall consider a linear form of the type $L(x) = x_1 + \dots + x_n$, $n \geqslant 2$. Any linear form with nonzero coefficients can be reduced to this type by a substitution $x_j' = \alpha_j x_j$. Let P(x) be a polynomial of degree m with real coefficients. The polynomial P is called admissible if at least one term x_j^m appears in the irreducible expression of P with a nonzero coefficient. We denote by d_k $(1 \leqslant k \leqslant m)$ the sum of the coefficients of terms of degree k in the polynomial P. Without loss of generality we can assume that the constant term in P vanishes.

THEOREM 1. Let x be a sample with replacement, and P be an admissible statistic of degree m such that one of the numbers $d_k \neq 0$. If P(x) and L(x) are independent random variables, then P(x) = const almost surely (a.s.).

The case of $d_k = 0$ for all k has been successfully investigated only with m = 3. Let P be a third-degree polynomial. We put

$$P(x) = \sum_{i,j,k=1}^{n} c_{ijk}x_ix_jx_k + \sum_{i,j=1}^{n} c_{ij}x_ix_j + \sum_{j=1}^{n} c_jx_j,\quad c_{ijk},\ c_{ij},\ c_j \in \mathbf{R}, \tag{1}$$

and we introduce the notation $a_1 = \sum_i c_{iii}, a_2 = \sum_{i\neq j}(c_{iij} + c_{iji} + c_{jii})$, $a_3 = \frac{1}{6}\sum_{i<j<k} c_{ijk}$, $a_4 = \sum_j c_{jj}$, $a_5 = \sum_{i\neq j} c_{ij}$, $a_6 = \sum_j c_j$.

THEOREM 2. Let x be a sample with replacement and P(x) be an admissible statistic of the the form (1) with at least one of the numbers $a_j \neq 0$. If P(x) and L(x) are independent random variables, then either x is a normal vector, or P(x) = const a.s.

Proof of Theorem 1. We denote by f the characteristic function (c.f.) of the distribution function F of the random variables x_j. From the independence of P(x) and L(x), and from Theorem 8.12 in [2], in view of the polynomial P being admissible, it follows that f is an entire function of a finite order. Using again the independence, and arguing as in the proof of a theorem in [2] (Lemma 8.3.1), we obtain a differential equation

$$\sum a_{j_1\dots j_n}\frac{f^{(j_1)}}{f}\dots\frac{f^{(j_n)}}{f} = A, \tag{2}$$

where the summation is extended to all sets of indices such that $j_1 + \dots + j_n \leqslant m$, and $a_{j_1\dots j_n}$ and A are some constants satisfying $\sum_{j_1+\dots+j_n=k} a_{j_1\dots j_n} = (\sqrt{-1})^k d_k,\ k = 1,\dots,m.$

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Applying the Wyman–Valiron method ([3], Chap. V), we use the formula $f^{(j)}(\zeta)/f(\zeta) = (1 + o(1))(\nu(r)/\zeta)^j$, $j = 1, 2, \ldots$. Here $r = |\zeta| \to \infty$, disregarding a set of finite logarithmic measure, and ζ is the point at which f attains its maximum module on the circle $|z| = r$, while $\nu(r)$ is the central index ([3], (8)). Substituting this formula in (2) we find $\sum_{k=1}^{m} ((\sqrt{-1})^k d_k + o(1))(\nu(r)/\zeta)^k = A$, $r \to \infty$, and owing to the conditions of the theorem, at least one of the numbers $d_k \neq 0$. It follows that $\nu(r) = O(r)$ as $r \to \infty$, and consequently the function f is of the exponential type.

Now it follows from Theorem 2.2.2 in [2] that the set of points of growth of F(t) is bounded, i.e., that the x_j are a.s. bounded. The proof is completed by applying the following lemma:

LEMMA. Assume that the random variables x_j in a sample with replacement are a.s. bounded at least on one side, and that an arbitrary statistic P(x) does not depend on L(x). Then P(x) = const a.s.

This lemma is proved by an argument analogous to one adduced in [1].

Proof of Theorem 2. In view of Theorem 1, we can assume that $d_k = 0$ $(k = 1, 2, 3)$. Equation (2) with $m = 3$ takes the form

$$i\left(a_1 \frac{f'''}{f} + a_2 \frac{f''f'}{f \cdot f} + a_3\left(\frac{f'}{f}\right)^3\right) + a_4 \frac{f''}{f} + a_5\left(\frac{f'}{f}\right)^2 = A, \quad i = \sqrt{-1}, \; A \in \mathbf{R} \tag{3}$$

the constant A being the expectation of P(x) taken with the opposite sign. Since the statistic P is admissible, the solution of this equation ought to be an entire function of finite order. Putting $w = f'/f$, we get $i(a_1 w'' + (3a_1 + a_2)w'w) + a_4 w' = A$. Here we used the fact that $a_6 = d_1 = 0$, $a_4 + a_5 = d_2 = 0$, $a_1 + a_2 + a_3 = d_3 = 0$. Integrating and multiplying by $-i$, we obtain the Riccati equation

$$a_1 w' + \frac{1}{2}(3a_1 + a_2) w^2 - i a_4 w = -iAz + C, \quad A \in \mathbf{R}, \; C \in \mathbf{C}. \tag{4}$$

We distinguish several cases: 1. $a_1 \neq 0$, $3a_1 + a_2 \neq 0$, $A \neq 0$. We shall show that in this case the equation (3) cannot have entire characteristic solutions. By means of the substitution $w = \frac{2a_1}{3a_1 + a_2} y + \frac{i a_4}{3a_1 + a_2} = \alpha y + \beta$ we reduce (4) to the form

$$y' + y^2 = iA_1 z + C_1, \quad A_1 \in \mathbf{R}, \; C_1 \in \mathbf{C}. \tag{5}$$

It is well known that all the solutions of this equation are meromorphic functions with an infinite number of poles, all the residua being equal to 1. Therefore, $y = v'/v$, where v is some entire function. Obviously, $f(x) = (v(z))^{\alpha} \exp \beta z$. For the function v we have the equation

$$v'' = (iA_1 z + C_1) v, \quad A_1 \in \mathbf{R}, \; C_1 \in \mathbf{C}. \tag{6}$$

This equation reduces to Airey's equation ([4], No. 23.4). Any solution of the equation (6) is known to be an entire function of completely regular growth of order 3/2. Consequently, the Phragmen–Lindelöf indicator $h(\theta)$ of the function f coincides with the indicator of the function v and, by the property of the ridge of an entire characteristic function f, satisfies the conditions

$$h(\theta) \leqslant h\left(\frac{\pi}{2}\right)(\sin\theta)^{3/2}, \qquad 0 \leqslant \theta \leqslant \pi, \tag{7}$$

$$h(\theta) \leqslant h\left(-\frac{\pi}{2}\right)|\sin\theta|^{3/2}, \qquad \pi \leqslant \theta \leqslant 2\pi. \tag{8}$$

We are going to show that the indicator of the solution of (5) cannot have the properties (7) and (8). Let, e.g., $A_1 > 0$. From the asymptotic relations adduced in [4] it follows that Eq. (6) has two linearly independent solutions, v_1 and v_2, with respective indicators

$h_1(\theta)=\varkappa\cos\left(\frac{3}{2}\left(\theta+\frac{\pi}{6}\right)\right)$, $-\frac{\pi}{2}\leqslant\theta\leqslant\frac{3\pi}{2}$, $h_2(\theta)=-\varkappa\cos\left(\frac{3}{2}\left(\theta+\frac{\pi}{6}\right)\right)$, $-\frac{7\pi}{6}\leqslant\theta\leqslant\frac{5\pi}{6}$, $\varkappa>0$. For any solution v of equation (6) we have $v = \gamma_1 v_1 + \gamma_2 v_2$ (γ_1 and γ_2 being constants). If $\gamma_1 = 0$, then $h_2(\theta) = h(\theta)$; if $\gamma_2 = 0$, then $h(\theta) = h_1(\theta)$; if $\gamma_1\gamma_2 \neq 0$, then $h(\theta)=\left|\varkappa\cos\frac{3}{2}(\theta+\pi/6)\right|$ in the neighborhood of the point $\theta = -\pi/2$. All the three cases are incompatible with (8). The case of $A_1 < 0$ is treated similarly; we then obtain a contradiction with (7).

2. $\alpha_1 \neq 0$, $3\alpha_2 + \alpha_2 \neq 0$, $A = 0$. Repeating the arguments of case 1 as far as Eq. (6), we find $v'' = C_1 v$. Hence $v(z)=\gamma_1\exp(\sqrt{C_1}z)+\gamma_2\exp(-\sqrt{C_1}z)$. Consequently, the function $f(z) = (v(z))^{\alpha}\exp\beta z$ is of the exponential type. Therefore, the random variables x_j are a.s. bounded, and it follows from the lemma that $P(x) = \text{const}$ a.s.

3. $\alpha_1 \neq 0$, $3\alpha_1 + \alpha_2 = 0$. Equation (4) takes the form $\alpha_1 w' - i\alpha_4 w = iAz + C$. The general solution of this equation is $w=C_1\exp\left(\frac{a_4}{a_1}iz\right)+Q(z)$, Q being a polynomial. If $C_1\alpha_4 \neq 0$, then $f(z)=\exp\int w(z)\,dz$ is a function of an infinite order. This contradicts Theorem 8.12 in [2]. If $C_1\alpha_4 = 0$, then f(z) is an entire function without zeros, and according to Theorem 3.13 in [2] the random variables x_j are normal.

4. $\alpha_1 = 0$. Equation (4) takes the form

$$1/2a_2w^2-ia_4w=-iAz+C. \tag{9}$$

If the values of the coefficients are such that (9) are meromorphic solutions, then w is a polynomial of the first degree, and we return to the normal distribution.

Hence the proof is complete.

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