# Problem 3, p. 157.

March 23, 2021

The problem asks the following: To solve the heat equation

$$u_t = k\Delta u$$

in the cylinder $r \le 1$ with the boundary conditions:

$$\lim_{z\to\pm\infty} u_z(r,\theta,z,t) = 0 \tag{1}$$

(this is how "insulated ends" has to be interpreted), and one of the following:

$$a) \qquad u_r(1,\theta,z,t) = 0, \tag{2}$$

or

$$b) \qquad u(1,\theta,z,t) = 0. \tag{3}$$

*Solution.*

First we separate time variables from space variables: $u(x,t) = T(t)S(x)$. We obtain

$$\frac{T'}{kT} = \frac{\Delta S}{S} = -\lambda^2,$$

so

$$T(t) = e^{-k\lambda^2 t}, \tag{4}$$

where $\lambda^2$ is an eigenvalue of the Laplacian.

For the space variables, we have in cylindrical coordinates:

$$S_{rr} + \frac{1}{r}S_r + \frac{1}{r^2}S_{\theta\theta} + S_{zz} + \lambda^2 S = 0.$$

Writing $S = R(r)\Theta(\theta)Z(z)$ we first separate $Z$:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda^2 = -\frac{Z''}{Z} = \mu. \tag{5}$$

So for $Z$ we have $Z'' + \mu Z = 0$. The general solution can be either an affine function (when $\mu = 0$, $Z(z) = c_1 z + c_2$) or a combination of sine and cosine (when $\mu > 0$), or a combination of exponentials (when $\mu < 0$), but in any case the boundary condition (1) implies that $Z'(z) \to 0$ when $z \to \pm\infty$, and this is only possible when $\mu = 0$ and $Z$ is constant. Since the initial condition is also independent of $z$, we may forget about $Z$: our solutions are independent on $z$.

With $\mu = 0$, equation (5) becomes

$$r^2 \frac{R''}{R} + r\frac{R'}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = m^2$$

so we separated the $\theta$, and since the boundary conditions for $\Theta$ are $2\pi$-periodic, we conclude that $m$ must be an integer, and

$$\Theta_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta), \quad m = 0, 1, 2, \ldots.$$

Now for the $r$-part we obtain

$$r^2 R'' + rR' + (\lambda^2 r^2 - m^2)R,$$

which is reduced to Bessel's equation (see equations (5.1) and (5.2) in the book), and since $R(0)$ must be finite, the solution must be a constant times $J_m(\lambda r)$.

Now we use one of the boundary conditions (2) or (3). First of them gives

$$a) \qquad J'_m(\lambda_{m,n}) = 0, \quad n = 1, 2, 3, \ldots, \tag{6}$$

and the second gives

$$b) \qquad J_m(\lambda_{m,n}) = 0, \quad n = 1, 2, 3, \ldots. \tag{7}$$

In other words, $\lambda_{m,n}$ are non-negative zeros of derivative of $J_m$ in case a), and non-negatve zeros of $J_m$ in case b).

So, the general solution satisfying boundary conditions in both cases is given by the formula

$$u(r, \theta, z, t) = \sum_{m,n} (a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta)) e^{-k\lambda_{m,n}^2 t} J_m(\lambda_{m,n} r), \tag{8}$$

where $\lambda_{m,n}$ have different meanings: for case a) they are zeros of $J'_m$, while for case b) they are zeros of $J_m$.

It remains to satisfy the initial condition. For this we plug $t = 0$ and write the initial condition in cylindrical coordinates:

$$ax + b = ar\cos\theta + b.$$

We conclude that the series (8) contains only terms with $m = 0$ or 1, and $b_m = 0$ for all $m$.

In case a)

$$ar\cos\theta + b = \cos\theta \sum_n a_n J_1(\lambda_{1,n} r) + b,$$

where we used that $\lambda_{0,1} = 0$, and $J_0(0) = 1$. Then Fourier-Bessel formulas give

$$a_n = \frac{2\lambda_{1,n}^2}{(\lambda_{1,n}^2 - 1)J_1^2(\lambda_{1,n})} \int_0^1 arJ_1(\lambda_{1,n} r) r dr.$$

Here we used Theorem 5.3 b) with $c = 0$ and $\nu = 1$. In evaluation of the integral we follow Example 2 on p. 155. By formula (5.14) with $\nu = 1$ we have

$$x^2 J_1(x) = \frac{d}{dx}(x^2 J_2(x)),$$

so the integral is equal to $J_2(\lambda_{1,n})/\lambda_{1,n}$, and we obtain

$$a_n = \frac{2a\lambda_{1,n} J_2(\lambda_{1,n})}{(\lambda_{1,n}^2 - 1)J_1^2(\lambda_{1,n})},$$

which matches the answer in the book.

And in case b) we similarly obtain:

$$ar\cos\theta + b = \cos\theta \sum_{n=1}^{\infty} a_{1,n} J_1(\lambda_{1,n} r) + \sum_{n=1}^{\infty} a_{0,n} J_0(\lambda_{0,n} r).$$

So by Fourier–Bessel formulas (Theorem 5.3 a)), we obtain

$$a_{1,n} = \frac{2a}{J_2^2(\lambda_{1,n})} \int_0^1 J_1(\lambda_{1,n} r) r^2 dr = \frac{2a}{\lambda_{1,n} J_2(\lambda_{1,n})},$$

and

$$a_{0,n} = \frac{2b}{J_1^2(\lambda_{0,n})} \int_0^1 J_0(\lambda_{0,n} r) r dr = \frac{2b}{\lambda_{0,n} J_1(\lambda_{0,n})}.$$

In the evaluation of these integrals, we applied the same method as in part a) (following Example 2 on p. 155), using Theorem 5.3 and formula (5.14).