A problem of Stanisław Saks

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Abstract

A solution of Problem 184 from the Scottish Book is given.

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On February 8, 1940, the following entry was made in the Scottish book [3]:

184. Problem; S. Saks.
A subharmonic function ϕ has everywhere partial derivatives $\partial^2\phi/\partial x^2,\ \partial^2\phi/\partial y^2$. Is it true that $\Delta\phi \geq 0$?
Remark: it is obvious immediately that $\Delta\phi \geq 0$ at all points of continuity of $\partial^2\phi/\partial x^2,\ \partial^2\phi/\partial y^2$, therefore on an everywhere dense set.
Prize: one kilo of bacon.

Theorem. *Let u be a subharmonic function of two variables whose first partial derivatives exist on the coordinate axes and $u_{xx},\ u_{yy}$ exist at the origin. Then $u_{xx}(0,0) + u_{yy}(0,0) \geq 0$.*

Proof. Without loss of generality we assume that $u(0,0) = u_x(0,0) = u_y(0,0) = 0$ (add a linear function). Proving the Theorem by contradiction, we assume that $\Delta u(0,0) < 0$. Then there exist real a, b and $R_0 > 0$ such that for $x^2 + y^2 < R_0^2$ we have

$$u(x,0) \leq ax^2, \quad u(0,y) \leq by^2, \quad \text{where} \quad a + b < 0. \tag{1}$$

Without loss of generality, $a < 0$.

If $b < 0$, consider the function

$$v_1(r\cos\theta, r\sin\theta) = Cr^2|\sin(2\theta)|,$$

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which is harmonic in each quadrant, and choose $C > 0$ so large that $v_1(x, y) \geq u(x, y)$ when $x^2 + y^2 = R_0^2$. Then $u(x, y) \leq v_1(x, y)$ for $x^2 + y^2 < R_0^2$ by the Maximum principle applied to the intersection of this disk with each quadrant. Thus

$$u(x, y) \leq C(x^2 + y^2), \quad \text{when} \quad x^2 + y^2 < R_0^2. \tag{2}$$

Consider the family of subharmonic functions

$$u_r(x, y) = r^{-2} u(rx, ry), \quad r > 0$$

In view of (2), for every compact K in the plane there exists $r_0 > 0$ such that u_r are defined and uniformly bounded from above on K for $r \in (0, r_0)$. Therefore there is a sequence $r_j \to 0$ for which $u_{r_j} \to u_0$ in L^1_{loc}, where u_0 is a subharmonic function, [1, Theorem 3.2.12]. Moreover $u(x, y) \geq \limsup_{r \to 0} u_0(x, y)$ for every x, y by [1, Theorem 3.2.13], so $u_0(0, 0) = 0$. To show that u_0 satisfies (1), fix a point $(x_0, 0)$, and consider disks B_t of radii t centered at this point. Since the family $\{u_r\}$ is uniformly bounded from above on B_1, there is a continuous majorant v for this family in B_1, such that $v(x_0, 0) \leq a x_0^2$. This v is just the solution of the Dirichlet problem for upper and lower halves of B_1 with boundary conditions ax^2 on the intersection of B_1 with the x-axis, and constant on the half-circles. So for every $\epsilon > 0$ there exists δ such that $v(x_0, 0) \leq a x_0^2 + \epsilon$ in B_δ. Then L^1_{loc} convergence gives

$$u_0(x_0, 0) \leq \frac{1}{|B_\delta|} \int_{B_\delta} u_0(x, y) dx dy \leq \frac{1}{|B_\delta|} \int_{B_\delta} v(x, y) dx dy \leq a x_0 + \epsilon.$$

As ϵ is arbitrary, we obtain that u_0 satisfies the first inequality in (1) on the whole x-axis. Similar arguments show that u_0 satisfies the second inequality in (1) on the whole y-axis, and also satisfies (2) in the whole plane.

The Phragmén–Lindelöf indicator of u_0,

$$h(\theta) := \limsup_{r \to \infty} r^{-2} u_0(r \cos \theta, r \sin \theta)$$

is non-positive for $\theta = \pi/2$ and negative for $\theta = 0$. This contradicts the inequality

$$h(\theta) + h(\theta + \pi/2) \geq 0,$$

which the indicators of all functions of order 2 must satisfy, [2, Section 8.2.4].

If $b \geq 0$, we consider the subharmonic function

$$u^*(x,y) = u(x,y) + c(x^2 - y^2),$$

where $b < c < -a$. Such a c exists because $a + b < 0$ in (1). Then u^* satisfies

$$u^*(x,0) \leq (a+c)x^2, \quad u^*(0,y) \leq (b-c)y^2$$

near the origin, and we apply the previous argument to u^*. This completes the proof.

Corollary. There is no subharmonic function u satisfying

$$u(0) = 0 \quad \text{and} \quad u(x,0) \leq -\epsilon|x|$$

for all sufficiently small x and $\epsilon > 0$.

Remark. The Theorem does not hold in R^n for $n \geq 3$. Indeed, in this case the union of the coordinate axes is a polar set, so it is easy to construct a counterexample.

References

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[3] R. D. Mauldin, The Scottish Book, Springer, NY, 2015. Online version of the English translation by S. Ulam, http://kielich.amu.edu.pl/Stefan_Banach/pdf/ks-szkocka/ks-szkocka3ang.pdf

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