Midterm exam solutions and comments

1. The boundary conditions imply that the eigenvalues are non-negative integers $n = 0, 1, 2\dots$ and eigenfunctions are $\cos nx$. We have to expand $\sin x$ in terms of those eigenfunctions. We have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos nx\,dx \\ &= \frac{1}{\pi}\int_0^\pi \sin(n+1)x + \sin(1-n)x\,dx \\ &= -\frac{1}{\pi}\left(\frac{1}{n+1}\cos(n+1)x + \frac{1}{1-n}\cos(1-n)x\right)|_0^\pi \\ &= -\frac{1}{\pi}\left(\frac{(-1)^{n+1}-1}{n+1} + \frac{(-1)^{1-n}-1}{1-n}\right) \\ &= -\frac{4}{\pi}\frac{1}{n^2-1} \end{aligned}$$

when n is even and positive. When n is odd we get zero. The zeroth coefficient is the average of sin that is $2/\pi$. So the solution of our problem is

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} e^{-4k^2 t}\frac{\cos 2kt}{4k^2-1}.$$

Of course, I am never sure what $\sin A \cos B$ really is:-) So I compute:

$$\begin{aligned} \sin A \cos B &= \frac{1}{4i}(e^{iA} - e^{-iA})(e^{iB} + e^{-iB}) \\ &= \frac{1}{4i}(e^{i(A+B)} - e^{-i(A+B)} + e^{i(A-B)} - e^{i(B-A)} \\ &= \frac{1}{2}(\sin(A+B) + \sin(A-B)). \end{aligned}$$

To answer the second question, you say that this solution tends to a constant (namely $2/\pi$) as $t \to \infty$. The temperature will be close to this constant within 1% when the rest of the series makes less than 1% of $2/\pi$. Let us see when the first term of the series will be less than that, that is when

$$(4/\pi)e^{-4t} < 2/(100\pi),$$

that is $t > (1/4)\ln 200$. We need $\ln 200$. We know that $3^4 = 81 < 200$, so $\ln 200 > 4$. On the other hand $\ln 200$ is surely less than 8, because $2.5^8 > 6^4 = 36^2 > 900$. So $4 < \ln 200 < 8$ and the time needed is between 1 and 2 seconds.

We neglected all terms of the series after the first one, because their size is negligible in comparison with the first term, for example, the second term is less that $(4/\pi)e^{-16}$ after one second of time, and so on.

2. a) f is real means that $f = \overline{f}$, that is

$$\sum c_n e^{inx} = \sum \overline{c_n} e^{-inx} = \sum \overline{c_{-n}} e^{inx},$$

and by uniqueness of the expansion we conclude that this happens if and only if $c_n = \overline{c_{-n}}$ for all n.

b) f is even means that $f(x) = f(-x)$. That is

$$\sum c_n e^{inx} = \sum c_n e^{-inx} = \sum c_{-n} e^{inx},$$

which means that $c_n = c_{-n}$ for all n.

c) f has period π means that $f(x+\pi) = f(x)$ for all x, that is

$$\sum c_n e^{inx} = \sum c_n e^{in(x+\pi)} = \sum c_n (-1)^n e^{inx},$$

which means that $c_n = (-1)^n c_n$ that is $c_n = 0$ for all odd n.

3. a) Yes. You could refer to any theorem proved in class. For example, to state that such operator with separated boundary conditions is always self-adjoint. “Such operator”means here that it is of the form $(ry')' + py$, where $p = 0$ and $r = 1$ are two real functions and r is positive. You could also refer to Lagrange's identity (with $r = 1$) which implies that if $f'\overline{g} - f\overline{g}'$ vanishes for all f and g satisfying the boundary conditions then these boundary conditions are self-adjoint.

b) Let $\nu = \sqrt{\lambda}$. Then the general solution of the equation is $a\cos\nu x + b\sin\nu x$, and the condition on the left end gives $a = 0$. Now the condition on the right end says that

$$\sin\nu + \nu\cos\nu = 0,$$

that is $\tan\nu = -\nu$. Notice that I asked about λ's between 0 and 2π, which corresponds to ν's between 0 and $2\sqrt{\pi}$. To count the solutions one has to draw the graph of $\tan\nu$ together with the graph of $-\nu$ and see where they intersect. When you draw the graphs correctly, you will find that they intersect once on the interval $(\pi/2, 3\pi/2)$ and then once on $(3\pi/2, 5\pi/2)$ and so on. Moreover, these intersection points lie on the left half of each of these intervals. Now $2\sqrt{\pi}$ lies in the right half of the first interval. (Indeed, $2\sqrt{\pi} > \pi$ because $\sqrt{\pi} < 2$, because $\pi < 4$. On the other hand, $2\sqrt{\pi} < 3\pi/2$, because $\sqrt{\pi} > 4/3$, because $\pi > 2 > 16/9$.) So we conclude that there is exactly one eigenvalue satisfying the condition of the problem.

4. The rule here is to expand everything into a series of eigenfunctions $\sin nx$, with coefficients depending on t. The initial condition was already expanded for you. Now we have to expand e^{-t}. Of course, *as a function of* x this is a *constant*, so we have to expand a constant, for example 1, into a sine-Fourier series. Computing the integrals,

$$\frac{2}{\pi}\int_0^\pi \sin nx dx = -\frac{2}{\pi n}(\cos nx)|_0^\pi = \frac{2}{\pi n}(1-(-1)^n).$$

This is zero when n is even and $4/\pi n$ when n is odd. Notice that the expansion of the initial condition given in the problem also has this property: only odd terms are present. This permits to simplify the computation by considering only odd n. Anyway, we have

$$e^{-t} = \frac{4}{\pi}e^{-t}\sum_{n \text{ odd}} \frac{\sin nx}{n} = \frac{4}{\pi}e^{-t}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}. \tag{1}$$

Now we look for a solution of our equation in the form

$$u(x,t) = \sum_{n \text{ odd}} c_n(t)\sin nt.$$

Substituting this and (1) to our equation, we obtain for each odd n:

$$c_n'(t) = -2n^2 c_n(t) + \frac{4}{\pi n}e^{-t}. \tag{2}$$

These are first order ordinary differential equations. One needs initial conditions. They come from setting $t = 0$ and using the given expansion of the initial condition:

$$c_n(0) = 8/(\pi n^3) \quad \text{for odd} \quad n. \tag{3}$$

The solution of the ordinary differential equation (2) above consists of two parts: the general solution of the homogeneous equation $ae^{-2n^2 t}$ and some particular solution of the non-homogeneous equation. Because of the very simple nature of the non-homegeneous term, constant times e^{-t}, it is reasonable to look for the particular solution in the form be^{-t}, where b is a constant. By substituting this form to (2) we find that

$$b = \frac{4}{\pi n(2n^2 - 1)}.$$

Now we find a from the initial condition (3):

$$c_n(0) = a + b = \frac{8}{\pi n^3}$$

so

$$a = \frac{8}{\pi n^3} - \frac{4}{\pi n(2n^2 - 1)} = \frac{4(3n^2 - 2)}{\pi n^3(2n^2 - 1)}.$$

And now we put everything together:

$$c_n = ae^{-2n^2 t} + be^{-t} = \frac{4(3n^2 - 2)}{\pi n^3(2n^2 - 1)} e^{-2n^2 t} + \frac{4}{\pi n(2n^2 - 1)} e^{-t},$$

and the final answer is

$$u(x,t) = \sum_{n \text{ odd}} \left(\frac{4(3n^2 - 2)}{\pi n^3(2n^2 - 1)} e^{-2n^2 t} + \frac{4}{\pi n(2n^2 - 1)} e^{-t} \right) \sin nt.$$

Of course, I understand that this problem is a bit labor-consuming. I gave partial credit for various parts of it.

5. The function is odd and its Fourier series is the same as the sine-Fourier series of the function 1. This was found in the process of solving Problem 4, so I do not repeat the computation:

$$\frac{x}{|x|} = 1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}, \quad 0 < x < \pi.$$

Now we can use the Parseval identity either on the interval $(-\pi, \pi)$ or on the interval $(0, \pi)$, this does not matter. The Parseval identity says that the square of the norm of the function equals to the sum of the squares of the norms of the terms of its orthogonal expansion. (This is just a version of Pythagoras theorem with infinite sums). Using the norm

$$\|f\|^2 = \int_0^\pi |f(x)|^2 dx,$$

we find:

$$\|1\|^2 = \pi,$$

$$\begin{aligned}\|\sin nx\|^2 &= \int_0^\pi \sin^2 ndx \\ &= \frac{1}{2}\int_0^\pi (1-\cos 2nx)dx = \frac{\pi}{2}.\end{aligned}$$

Thus

$$\pi = \|1\|^2 = \frac{16}{\pi^2}\sum_{n=0}^\infty \frac{\|\sin(2n+1)x\|^2}{(2n+1)^2} = \frac{\pi}{2}\frac{16}{\pi^2}\sum_{n=0}^\infty \frac{1}{(2n+1)^2}$$

and the answer is

$$\sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

You could use, of course another system e^{inx} but remember: most orthogonal systems we use are not *orthonormal*, for example,

$$\|e^{inx}\|^2 = \int_{-\pi}^{\pi} e^{inx}e^{-inx}dx = 2\pi,$$

not 1!