MATH 520, Midterm exam, Spring 2011

NAME:

1. Suppose that f is a smooth 2π-periodic function, and

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{int}$$

its Fourier expansion. How can one find out from the sequence (c_n) that
a) f is real,
b) f is pure imaginary,
c) f is even,
d) f is odd,
e) f has π as a period.

2. Consider the Sturm–Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

a) Is this problem self-adjoint?
b) How many eigenvalues λ_j satisfy $0 < \lambda_j < 4\pi$?

3. Suppose that

$$xe^x = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad |x| < \pi.$$

Let

$$g(x) = \sum_{n=1}^{\infty} a_n \cos nx,$$

and

$$h(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Find a_0, g and h.

Hint: computing a_n and b_n for $n \geq 1$ is hard. There is a simple solution.

4. Solve the heat equation

$$u_t = u_{xx}, \quad 0 \le x \le \pi, t \ge 0,$$

with boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ (insulated ends) and initial condition $u(x,0) = 1 + \cos(3x), 0 < x < \pi$.

5. Given that the sine Fourier series for $f(x) = x(\pi - x), 0 \le x \le \pi$ is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin nx,$$

find the solution of the non-homogeneous wave equation

$$u_{tt} = 2u_{xx} + e^t, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0.$$