Midterm exam solutions

1. By the uniqueness of Fourier expansion we have:
a) $f = \overline{f}$ is equivalent to $c_n = \overline{c_{-n}}$.
b) $f = -\overline{f}$ is equivalent to $c_n = -\overline{c_{-n}}$.
c) $f(t) \equiv f(-t)$ is equivalent to $c_n = c_{-n}$.
d) $f(t) \equiv -f(-t)$ is equivalent to $c_n = -c_{-n}$.
e) $f(t) \equiv f(t+\pi)$ is equivalent to $c_n = (-1)^n c_n$, that is $c_n = 0$ for odd n.

2. Let $\lambda = \mu^2 > 0$. The general solution is $y(x) = a\cos\mu x + b\sin\mu x$, and the first condition implies that $a = 0$, and we can take $b = 1$. The second condition gives $\sin\mu + \mu\cos\mu = 0$ that is

$$\tan\mu = -\mu.$$

By graphing this, we see that there is exactly one solution on each of the intervals $(\pi/2 + k\pi, \pi)$, $(3\pi/2, 2\pi)$, $(5\pi/2, 3\pi)$ etc. The interval $0 < \lambda < 4\pi$ corresponds to $0 < \mu < 2\sqrt{\pi}$. It is clear that $\pi < 2\sqrt{\pi} < 3\pi/2$, so there is only one eigenvalue answering the question.

3. a_0 is the average:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^x dx = \frac{1}{2\pi}(xe^x)|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx \\ &= (e^\pi + \pi e^{-\pi})/2 + (e^\pi - e^{-\pi})/(2\pi) = \cosh\pi - \frac{\sinh\pi}{\pi}. \end{aligned}$$

Now $f(x) := xe^x - a_0 = g(x) + h(x)$, where g is even and h is odd, so

$$g(x) = (f(x) + f(-x))/2 = -a_0 + (xe^x - xe^{-x})/2 = -a_0 + x\sinh x,$$

and

$$h(x) = (f(x) - f(-x))/2 = (xe^x + xe^{-x})/2 = x\cosh x.$$

4. Separating the variable $u = XT$ we obtain

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

The Sturm Liouville problem

$$X'' + \lambda X = 0, \quad X'(0) = X'(\pi) = 0$$

has eigenvalues $\lambda = n^2,\ n = 0, 1, 2 \dots$ and eigenfunctions $\cos nx$. Solving

$$T' = -n^2 T$$

we obtain $T(t) = ae^{-nt}$. So the general solution satisfying the boundary conditions is

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-nt} \cos nx.$$

To satisfy the initial condition we put $t = 0$ and conclude that $a_0 = 1, a_3 = 1$ and the rest are 0, that is

$$u(x,t) = 1 + e^{-9t} \cos(3x).$$

5. Separating the variables in the homogeneous equation we obtain

$$\frac{T''}{2T} = \frac{X''}{X} = -\lambda.$$

The Sturm Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0$$

has eigenvalues $\lambda = n^2,\ n = 1, 2, \dots$ and eigenfunctions $\sin nx$. Now we look for the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin nx.$$

Using entry 6 from the Fourier expansion tables,

$$e^t = \frac{4e^t}{\pi} \sum_{1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

substituting all this to the equation we obtain for odd n:

$$c_n''(t) = -2n^2 c_n(t) + \frac{4e^t}{\pi n}, \tag{1}$$

with the initial conditions which come from the expansion of f:

$$c_n(0) = \frac{8}{\pi n^3}, \quad c_n'(0) = 0. \tag{2}$$

For even n we obtain

$$c_n''(t) = -2n^2 c_n(t), \quad c_n(0) = c_n'(0) = 0.$$

So for even n, $c_n \equiv 0$. For odd n, we solve the ordinary differential equations (1). The general solution is

$$c_n(t) = a_n \cos(n\sqrt{2}t) + b_n \sin(n\sqrt{2}t) + \frac{4e^t}{\pi n(2n^2+1)}.$$

(A partial solution of the non-homogeneous equation is easy to guess, or one can use any regular method to find it. For example, set $c_n(t) = Ae^t$ and substitute to (1). We get $A = 4/(\pi n(2n^2+1))$.) Now initial conditions give

$$a_n = \frac{8}{\pi n^3} - \frac{4}{\pi n(2n^2+1)} \tag{3}$$

and

$$b_n = -\frac{2\sqrt{2}}{\pi n^2(2n^2+1)}. \tag{4}$$

So the solution of our problem is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx) \left(a_n \cos(n\sqrt{2}t) + b_n \sin(n\sqrt{2}t) + \frac{4e^t}{\pi n^2(2n^2+1)} \right),$$

where a_n and b_n are as in (3) and (4).