MA 440

Final Examination

Fall 2012

NAME ____________________ ID number ____________________

In your proofs you may use any theorem (not an exercise!) in the textbook, but when you use it you must explicitly refer on it

Points awarded

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

8. __________

Total Points: __________

1. a) Prove that every sequence of real numbers contains a monotone subsequence.

b) Describe those sequences of real nmbers that do not contain a strictly monotone subsequence.

2. Let $g(x) = [x]$ be the floor function, namely $g(x)$ is the largest integer which is less than or equal to x. Evaluate the integral

$$\int_0^3 x dg.$$

3. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally (not absolutely) convergent series and let

$$y_n = \max\{x_n, 0\}.$$

Does it follow that the series $\sum_{n=1}^{\infty} y_n$ is absolutely convergent, conditionaly convergent, or divergent ?

Justify your answer.

4. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that there exists x_0 such that $f(x_0) = x_0$.

5. A limit point of a sequence is defined as a limit of a convergent subsequence. Prove that the set of all limit points of an arbitrary sequence is closed.

6. Let a sequence (a_n) of real numbers satisfy the recurrent relation

$$a_{n+1} = (a_n + 3)/4, \quad n \geq 0.$$

a) Prove that for every initial value a_0 the limit exists.

b) Find the limit.

7. Prove that for every convergent series

$$\sum_{n=0}^{\infty} a_n$$

with positive coefficients, one can find a sequence of positive numbers $b_n \to +\infty$ such that the series

$$\sum_{n=0}^{\infty} a_n b_n$$

is also convergent.

8. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function for which the derivative exists at all points. Is it possible that the right and left limits

$$\lim_{x \to 0+} f'(x) \quad \text{and} \quad \lim_{x \to 0-} f'(x)$$

both exist but are not equal?

Give an example or prove that this is impossible.