Math 520 midterm exam, spring 2021

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1. Let f be a continuous, piecewise-smooth function with Fourier expansion

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx, \quad 0<x<\pi.$$

Suppose that f satisfies

$$f(x)=f(\pi-x), \quad 0<x<\pi.$$

Find b_{2n}, for all $n=1,2,3,\ldots$. Justify your answer.

Solution. First method.

$$b_{2n}=\frac{2}{\pi}\int_0^\pi f(x)\sin(2nx)\,dx.$$

Changing the variable to $y=\pi-x$ and using the assumption about f, we obtain

$$b_{2n}=\frac{2}{\pi}\int_0^\pi f(y)\sin(2n(\pi-x))\,dx=-b_{2n}.$$

So $b_{2n}=0$.

Second method.

$$f(\pi-x)=\sum_{n=1}^{\infty} b_m \sin(n(\pi-x))=-\sum_{n \text{ even}} b_n\sin(nx)+\sum_{n \text{ odd}} b_n\sin(nx),$$

where we used that sine is odd and has period 2π. This must be equal to $f(x)$, so by uniqueness theorem, $b_n=-b_n$ when n is even. Thus $b_{2n}=0$.

2. Consider the Fourier expansion

$$e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \quad \pi < x < \pi.$$

a) Find a_0.

b) Find $\sum_{n=1}^{\infty}(-1)^n a_n$. (*Hint: this is the value of* $\sum_{n=1}^{\infty} a_n \cos nx$ *at* $x = \pi$.)

Solution. a) The constant term is the average:

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x dx = (e^{\pi} - e^{-\pi})/(2\pi) = (\sinh \pi)/\pi,$$

so $a_0 = 2(\sinh \pi)/\pi$.

b) Let

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

This is the even part of our function, and its 2π periodic extension is continuous, so

$$S(x) = (e^x + e^{-x})/2, \quad \text{and} \quad S(\pi) = (e^{\pi} + e^{-\pi})/2 = \cosh \pi.$$

and

$$\sum_{n=1}^{\infty} (-1)^n a_n = S(\pi) - \frac{a_0}{2} = \cosh \pi - (\sinh \pi)/\pi.$$

3. Which of the following sequences are Fourier coefficients of some function $f \in L^2(-\pi, \pi)$?

a) $c_n = \dfrac{n}{n^2+1}$,

b) $c_n = \dfrac{(-1)^n}{1+\sqrt{|n|}}$,

c) $c_n = \dfrac{1}{(1+\sqrt{|n|})\log(1+n^2)}$.

Solution. By the Riesz–Fisher Theorem, it is necessary and sufficient that

$$\sum_n |c_n|^2 < \infty.$$

For a) this condition is satisfied, since $|c_n|^2 \sim n^{-2}$.

For b) it is not satisfied since $|c_n|^2 \sim n^{-1}$ and

$$\sum_n \frac{1}{n} = \infty,$$

by comparison with the integral.

For c), it is satisfied since $|c_n|^2 \sim 4^{-1}n^{-1}(\log n)^{-2}$, and the series is convergent by comparison with the integral

$$\int^{\infty} \frac{dx}{x(\log^2 x)} = \int^{\infty} \frac{du}{u^2},$$

where $u = \log x$.

So the answer is: a) and c).

4. Consider the one-dimensional wave equation

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0,$$

and initial conditions

$$u(x,0) = x^2(1-x), \quad \text{and} \quad u_t(x,0) = 0.$$

Find $u(1/2, 3)$. (The answer should be an explicit number, not a series!).

Solution. Use d'Alembert's formula. Let $\phi(x) = x^2(1-x)$, and $\tilde{\phi}$ the 2**-periodic, odd extension of** $\phi(x) = x^2(1-x)$**.** d'Alembert's formula gives the solution in the form

$$u(x,t) = \frac{1}{2}\left(\tilde{\phi}(x+t) + \tilde{\phi}(x-t)\right),$$

so

$$u(1/2,3) = \frac{1}{2}\left(\tilde{\phi}(7/2) + \tilde{\phi}(-5/2)\right).$$

Now, since $\tilde{\phi}$ is 2**-periodic and odd**, we have

$$\tilde{\phi}(7/2) = \tilde{\phi}(7/2-4) = \tilde{\phi}(-1/2) = -\tilde{\phi}(1/2),$$

and

$$\tilde{\phi}(-5/2) = \tilde{\phi}(-5/2+2) = \tilde{\phi}(-1/2) = -\tilde{\phi}(1/2).$$

So

$$u(1/2,3) = -\tilde{\phi}(1/2) = -\phi(1/2) = -1/8.$$

PLEASE, RE-READ THE HANDOUT "D'ALEMBERT FORMULA" AS MANY TIMES AS NECESSARY, AND ASK QUESTIONS, UNTIL YOU UNDERSTAND IT!

5. For the Sturm-Liouville problem:

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = -y(1)/2,$$

answer the following questions:

a) Is it self-adjoint?

b) How many eigenvalues λ belong to the interval $(0, 25)$

c) Are there any negative eigenvalues? How many?

All answers have to be justified.

Solution. a) Yes. The boundary conditions are separated.

b) Let $\lambda = \omega^2$. Then ω is either real or pure imaginary, but in any case we have

$$y(x) = a \cos \omega x + b \sin \omega x,$$

$$y'(x) = -a\omega \sin \omega x + b\omega \cos \omega x.$$

Condition $y(0) = 0$ implies that $a = 0$. Condition $y'(1) = -y(1)/2$ gives

$$\tan \omega = -2\omega.$$

By sketching the graph for real ω, we see that this equation has two solutions on the interval $(0, 2\pi)$, and the bigger one is greater than $3\pi/2$. Since

$$3\pi/2 < 5 = \sqrt{25} < 2\pi,$$

we conclude that there are two eigenvalues λ on the interval $(0, 25)$. Remark. $(0, 25)$ means $0 < \lambda < 25$.

If $\lambda < 0$, then ω is pure imaginary, say $\omega = it$, then we have $\tan it = i \tanh t$, and the equation becomes

$$\tanh t = -2t.$$

Sketching the graph, we see that this equation has no solutions except 0. Zero is not called a negative number. There are no negative eigenvalues λ.