First Midterm solutions

Problem 1. a) Let E be a closed set. Put

$$A_n = \bigcup_{x \in E} \left\{ y \in \mathbf{R}^n : \|x - y\| < \frac{1}{n} \right\},$$

That is A_n is the union of open balls of radius $1/n$ with centers on E. Then each A_n is open (as a union of open sets), and each A_n contains E. Let us prove that

$$E = \bigcap_{n=1}^{\infty} A_n.$$

First E is contained in this union, because it is contained in each A_n. Now let us take a point $z \notin E$. By a theorem from the book, there is a positive number d (distance between x and E) such that $\|z - x\| \geq d$ for all $z \in E$. So $x \notin A_n$ for $n > 1/d$. Thus x is not in the intersection of A_n and the intersection equals E.

b) One example is $\mathbf{Q}$, the set of rational numbers. I did not ask for a proof in this part, but here is a proof. Proving the statement by contradiction, suppose that $\mathbf{Q} = \bigcap_{n=1}^{\infty} A_n$, where all A_n are open.

Let $x_1, x_2, \ldots$ be some enumeration of all rational numbers.

Inside A_1 we can find an closed interval I_1 (which is not a point) of length < 1, and which does not contain x_1. There are rational points inside every interval, and A_2 contains all of them, so we can find a smaller interval $I_2 \subset I_1 \cap A_2$, of length $< 1/2$, and which does not contain x_2.

Continuing this procedure, we find an interval I_n of length $< 2^{-n}$ inside $I_{n-1} \cap A_n$, and not containing x_n.

The intersection of these nested intervals I_n is one point. This point belongs to all A_n, by construction. But this point cannot be rational, because each rational number x_k was excluded on the k-th step.

Problem 2. $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ is the simplest example. For some other examples, like those involving $\sin n$, the proof is required.

Problem 3. a) By induction. Suppose we already know that

$$b < b_1 < \ldots < b_n < a_n < \ldots < a_1 < b$$

. Then $a_{n+1} = (a_n + b_n)/2 < a_n$, and

$$b_{n+1} = \left(\frac{1}{2}\left(\frac{1}{a_n} + \frac{1}{b_n}\right)\right)^{-1} > b_n.$$

And

$$a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \frac{2a_n b_n}{a_n + b_n} = \frac{(a_n + b_n)^2 - 4a_n b_n}{2(a_n + b_n)} > 0.$$

This proves a).

b) Using the previous formula,

$$a_{n+1} - b_{n+1} = \frac{(a_n - b_n)^2}{2(a_n + b_n)}.$$

From this follows that

$$|a_{n_1} - b_{n+1}| \leq \frac{1}{2}|a_n - b_n|.$$

So on each step, the difference decreases by at least a factor of 2, so $a_n - b_n \to 0$.

c) By a), a_n is decreasing and bounded from below; b_n is increasing and bounded from above, so both sequences have limits, and by b), the limits are equal.

d) We have

$$a_{n+1}b_{n+1} = \frac{a_n + b_n}{2}\frac{2a_n b_n}{a_n + b_n} = a_n b_n,$$

and the statement follows by induction.

e) Let x be the common limit of a_n and b_n. Then by d), $x.x = a.b$ so $x = \sqrt{ab}$.

Problem 4. The statement in the problem is wrong.

Here are two open sets:

$$A = \{(x, y) : x \in \mathbf{R},\ y > 0\} \quad \text{and} \quad B = \{(x, y) : x \in \mathbf{R},\ y < 0\}.$$

They are open, disjoint, and each contains points with both irrational coordinates, for example $(\pi, \pi) \in A$ and $(\pi, -\pi) \in B$. They also cover all points with both irrational coordinates. Indeed, let (a, b) be such point. Both a

and b are irrational. So $b \neq 0$. If $b > 0$, the point belongs to A, if $b < 0$ it belongs to B.

You should be able to tell yourself a wrong statement from a correct one, and even more importantly, you should be able to tell a proof from a "non-proof".

Problem 5. The simplest example is any sequence enumerating rational numbers. A sequence which contains each rational number once. But one needs a justification. Justification is simple. Let x be a real number. Then there exists a sequence of *distinct* rational numbers that tends to x. For example, a strictly increasing sequence of rational numbers. This sequence is a subsequence of the sequence of all rational numbers.

I've seen other examples in the exam papers. Some of them are wrong, some are correct. For the correct examples, proving that they have the required property is not simple, and it was never given. If your sequence is correct and you can *prove* this I will add you credit.