Operations on matrices

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August 22, 2024

I recall the notation: $\mathrm{Mat}(m \times n)$ stands for the set of all matrices with m rows and n columns. We say that such a matrix has *size* $m \times n$. When $n = 1$ a matrix is a column vector, and when $m = 1$ it is a row vector. When both $m = n = 1$, the matrix is simply a number. So matrices are generalization of numbers. This suggests an introduction of arithmetic of matrices.

Matrices *of the same size* can be added and multiplied by numbers. This is performed entry-wise. So one can consider linear combinations of matrices of the same size. For example:

$$2\begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 3 \\ 7 & 8 \end{pmatrix}.$$

A matrix whose all elements are 0 is called the zero matrix. There is one zero matrix of each size, and it is denoted by $\mathbf{0}_{m,n}$ or simply by $\mathbf{0}$ if the size is clear. We have $A + \mathbf{0} = A$, and $A + (-1)A = \mathbf{0}$, for every A of appropriate size.

Multiplication of matrices is more tricky.

First I explain how to multiply a row on a column (row on the left, column on the right): we multiply first entry of the row on the first entry of the column, second entry of the row on the second entry of the column, and so on, and add the results. So the result of multiplication is a number (1×1 matrix). For example

$$(2,3,5)\begin{pmatrix} 7 \\ 11 \\ 13 \end{pmatrix} = 2 \cdot 7 + 3 \cdot 11 + 5 \cdot 13 = 112.$$

In general, one can multiply a matrix $A \in \mathrm{Mat}(m \times n)$ on a matrix $B \in \mathrm{Mat}(n \times p)$. *Notice that the width of A is equal to the height of B; A stands on the left and B on the right, otherwise multiplication is not defined!* The result $C = AB$ will be of size $m \times p$. The rule is the following: to obtain the element $c_{i,j}$ of C one multiplies the i-th row of A on the j-th column of B by the above rule:

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}, \quad 1 \le i \le m, \quad 1 \le j \le p.$$

For example,

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 & 1 \\ -1 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 & 3 \\ -3 & 3 & -3 & 0 \\ -7 & 8 & -6 & 1 \end{pmatrix}.$$

Multiplication of matrices is **not** commutative. In the previous example the product BA is not defined! Even if both AB and BA are defined, they are in general not equal, for example, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 4 & 6 \\ 10 & 12 \end{pmatrix}, \quad \text{and} \quad BA = \begin{pmatrix} 2 & 4 \\ 10 & 14 \end{pmatrix}.$$

When it happens that $AB = BA$ we say that A and B *commute*.

Evidently $A \cdot \mathbf{0} = \mathbf{0}$ and $\mathbf{0} \cdot A = \mathbf{0}$. But unlike for numbers $AB = \mathbf{0}$ *does not* imply that one of the A, B is zero. For example

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \mathbf{0}.$$

Multiplication of matrices is *associative*:

$$A(BC) = (AB)C$$

and distributive

$$A(B + C) = AB + AC, \quad (A + B)C = AC + BC.$$

These equations assume that the matrices have appropriate sizes which allows addition and multiplication.

The role of the unity is played by the *unit matrix*

$$I = I_n = \left(\begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{array} \right).$$

Formally $a_{ii} = 1$ and $a_{i,j} = 0$ for $i \neq j$. The unit matrix is *square*, say of size $n \times n$. We have

$$AI_n = A \quad \text{and} \quad I_m A = A$$

for all $A \in \mathrm{Mat}(m \times n)$.

We say that a matrix A has an *inverse* or *is invertible* if there exists B such that

$$AB = I \quad \text{and} \quad BA = I.$$

We denote this by $B = A^{-1}$. We will later see that only square matrices can be invertible. We can also define a *right inverse* B_r and a *left inverse* B_ℓ by equations

$$AB_r = I \quad \text{and} \quad B_\ell A = I.$$

Theorem 1. *If A has both left and right inverses then they are equal.*

Proof. Start with $AB_r = I$ and multiply both sides from the left on B_ℓ. We obtain

$$B_\ell = B_\ell I = B_\ell(AB_r) = (B_\ell A)B_r = IB_r = B_r.$$

Exercise 1. Show that the 1×2 matrix $(1, 0)$ has a right inverse, but no left inverse. This right inverse has a left inverse, namely $(1, 0)$ but no right inverse.

We will later show that only square matrices can be invertible (that is only square matrices can have inverses from both sides (and these inverses if exist, are equal by Theorem 1).

Exercise 2 (harder) Show that $I + AB$ is invertible if and only if $I + BA$ is invertible (for any matrices A, B for which these expressions are defined).

(In this and similar statements we always assume that the sizes are appropriate so that all expressions are defined).

Theorem 2. *If A and B are invertible then AB is invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. Direct verification. Notice the reversion of the order!

Now we consider some special kinds of matrices.

Diagonal matrices. These are square matrices with $a_{ii} = d_i$ and $a_{i,j} = 0$ for $i \neq j$. Such a matrix is denoted by

$$\mathrm{diag}(d_1, d_2, \ldots, d_n).$$

Multiplication rule gives

$$\mathrm{diag}(d_1, d_2, \ldots, d_n)\mathrm{diag}(c_1, c_2, \ldots, c_n) = \mathrm{diag}(d_1c_1, d_2c_2, \ldots, d_nc_n).$$

So the product of two diagonal matrices is diagonal. We also see that diagonal matrices *commute*: $D_1D_2 = D_2D_1$.

The following rules are easy to check:

Multiplication by a diagonal matrix from the left results in multiplying the rows on d_j.

Multiplication by a diagonal matrix from the right results in multiplying columns on d_j.

For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} pa & qb \\ pc & qd \end{pmatrix}$$

and

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} pa & pb \\ qc & qd \end{pmatrix}.$$

Triangular matrices. A matrix is called *lower triangular* if $a_{i,j} = 0$ for $i < j$, that is all elements *above* the main diagonal are zero. *The product of two lower triangular matrices is lower triangular.* Indeed, let $A = (a_{i,j}), B = (b_{i,j})$ and $C = AB = (c_{i,j})$. By definition of multiplication,

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}. \tag{1}$$

Suppose that A and B are lower triangular, that is

$$a_{i,k} = 0 \quad \text{for} \quad i < k \tag{2}$$

and

$$b_{k,j} = 0 \quad \text{for} \quad k < j. \tag{3}$$

Now considers some $c_{i,j}$ with $i < j$. Then every k satisfies either $k < j$ or $k > i$, so either (2) or (3) applies, and we conclude that all summands in (1) are zeros, so $c_{i,j} = 0$.

Similarly we can define *upper triangular* matrices as those for which $a_{i,j} = 0$ for $i > j$, and prove that the *product of upper triangular matrices is always upper triangular.*

The entries $a_{i,i}$ of a matrix are called the *main diagonal*. Suppose that we have two lower triangular matrices whose main diagonals consist of 1's. Then the product of these two matrices has the same property: it is lower triangular, with 1's on the main diagonal. This is also an easy consequence of the multiplication rule.

The similar rule we have for upper triangular matrices.