Row operations and multiplication of matrices

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Let us begin with the unit matrix and apply to it the row Operation 1, we add a multiple of some row to a lower row. For example, add ℓ times the first row to the third row of a 3×3 unit matrix. We obtain

$$L_{3,1}(\ell) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell & 0 & 1 \end{pmatrix}.$$

In general, if for the unit matrix we add ℓ times the j-th row to the i-th row, we obtain the matrix whose main diagonal consists of 1's, the entry on (i,j) is ℓ and the rest of the entries are zero. Such a matrix is denoted by $L_{i,j}(\ell)$. If $i > j$ it is lower triangular; if $i < j$ it is upper triangular.

Now it is easy to see what is the result of multiplication of an arbitrary matrix A on $L_{i,j}(\ell)$ from the left: it performs on A the same row operation: adding ℓ times the j-th row to the i-th row. For example

$$L_{2,1}(2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c+2a & d+2b \end{pmatrix}.$$

Therefore, Operation 1 on rows can be obtained by multiplication on these special matrices from the left. Moreover, notice that in our algorithm of reduction of a matrix to a REF we added multiples of higher rows to lower rows, so this is performed by *lower triangular* matrices $L_{i,j}(\ell)$ with $i > j$. So, in the generic case, when no row exchanges are required, we can represent our algorithm of obtaining REF as a sequence of multiplications on those special lower triangular matrices:

$$L_k L_{k-1} \ldots L_2 L_1 A = U,$$

where U is a REF. Notice that U is *upper triangular*, by the definition of the row echelon form.

Now, we can simply *invert* a row Operation 1: what we added can be subtracted to restore the original matrix A. This suggests that the matrices $L_{i,j}(\ell)$ are *invertible* and the inverse is given by the formula

$$(L_{i,j}(\ell))^{-1} = L_{i,j}(-\ell).$$

This is easy to verify.

So we have

$$A = (L_1)^{-1}(L_2)^{-1} \dots (L_k)^{-1}U =: LU,$$

where L is lower triangular and U is upper triangular. Notice the reverse order when we invert matrices!

So we obtain a factorization of a generic matrix

$$A = LU$$

into a lower and upper triangular factor. "Generic" means that no row exchanges are required in bringing A to REF.

If all rows of U contain pivots, we can go further and write

$$A = LDU,$$

where D is a diagonal matrix whose diagonal consists of pivots, and all left-most non-zero elements of U are equal to 1. Notice that on the main diagonal of L all elements are 1.

We will later prove that for a given A, if a factorization $A = LDU$, if possible, is unique.

Exchange and permutation matrices. We already know that Operation 1 on the rows can be performed by multiplication on an $L_{i,j}(\ell)$ from the left, and Operation 3 (multiplication of rows by constants) can be performed by multiplication on a diagonal matrix from the left. It remains to address the row exchange operation (Operation 2).

We follow the same argument as before. Let E be the result of a row exchange of the unit matrix. Then multiplication on this E from the left results in the exchange of the same rows on an arbitrary matrix. For example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Such a E is called an *exchange matrix*. If we make several exchanges consequently we obtain some *permutation* of the rows; just a reordering of the rows in some new order. Accordingly a matrix obtained by a permutation of the rows of the unit matrix is called a *permutation matrix*.

Notice that an exchange matrix is invertible, moreover it is equal to its own inverse $E = E^{-1}$. It follows that all permutation matrices are invertible, as products of exchange matrices. Permutation matrices are characterized by these properties:

a) It is a square matrix whose each column contains exactly one entry equal to 1, the rest of the entries re zeros, and

b) in each pair of columns these 1's stand in different positions.

Same is true about the rows of permutation matrices.

Exercise. Describe what happens if we multiply a matrix A on a permutation matrix from the left.

Ans.: It permutes rows.

Returning to reduction a matrix to REF, in general we have to perform a sequence of operations 1 and exchanges (operations 3), so we have

$$L_k E_q L_{k-1} E_{q-1} \dots L_1 A = U \tag{1}$$

or something similar. To simplify this, consider a pair EL in this product.

Claim. *Let $L = L_{i,j}(\ell)$ where $i > j$ be an elementary row operation matrix ($i > j$ means that it adds to a lower row i a multiple of a higher row j), and $E = E_{p,q}$ the matrix which exchanges p and q, where $p > j$ and $q > j$. Then*

$$EL = L_1 E \quad \text{which is the same as} \quad L_1 = ELE$$

where L_1 is a lower triangular matrix. (In fact L_1 is a matrix of the same kind as L; it is obtained from L by permutation of rows p, q and columns p, q.)

Proof. We know that P is invertible and $E = E^{-1}$. So we have $L_1 = ELE$. Multiplication on P from the left interchanges rows p, q, while multiplication on P from the right interchanges columns p, q. Since L is lower triangular, and its only non-zero element away from the main diagonal is at the place (i, j) with $i > j$, and $p, q > j$, this element either is not affected by our operation, or if $i \in \{p, q\}$ is moved to another place (with the same j and $i \in \{p, q\}$, still below the main diagonal.) Make a picture to see what happens!

Using this claim, we can move all E towards A in the sequence (1), so that new ‘matrices L remain lower triangular. The final result is

$$PA = LU, \tag{2}$$

where P is permutation matrix, L is lower triangular, and U is upper triangular. And such a factorization is possible *for all* matrices.

We do not discuss an efficient algorithm for doing $PA = LU$ factorization. In practice you can do Operations 1 and row exchanges as needed, then combine all row exchanges to make one permutation matrix, multiply A on it, and then do Operations 1 from scratch. No row exchanges will be required. Since you will do this exercise for matrices of size at most 3 or 4, this should not cause any difficulty. In all these calculations, pay attention to the order: exchange matrices in general do not commute!