Extremal Principles

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Let A and M be symmetric matrices, with M *positive definite*. Then the corresponding quadratic forms can be simultaneously reduced to linear combinations of squares, which means that there exists a non-singular matrix C such that

$$C^T MC = I, \quad C^T AC = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{1}$$

where λ_j are *generalized eigenvalues* solving the generalized characteristic equation $\det(A - \lambda M) = 0$.

All eigenvalues are real, and we label them in the non-decreasing order, each eigenvalue is repeated according to its multiplicity so that:

$$\lambda_1 \leq \lambda_2 \ldots, \leq \lambda_n.$$

The purpose of the following is to give formulas of the generalized eigenvalues which do not depend on any choice of coordinates.

Let us introduce the function, called the *Rayleigh ratio*, which is defined for $\mathbf{x} \neq 0$:

$$R(\mathbf{x}) = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T M\mathbf{x}}, \quad \mathbf{x} \neq 0.$$

Our goal is to study its extremal properties. Notice that R is homogeneous, $R(k\mathbf{x}) = R(\mathbf{x})$ for any real k. It follows that it has a maximum and a minimum.

Consider the transformation $\mathbf{x} = C\mathbf{y}$, where C is the non-singular matrix from (1). We have $\mathbf{y} = C^{-1}\mathbf{x}$, so the coordinates y_k of $\mathbf{y}$ are $y_k = \mathbf{u}_k{}^T\mathbf{x}$, where $\mathbf{u}_k^T$ is the k-th row of C^{-1}. We also denote by $\mathbf{v}_k$ the k-th column of C, so that

$$\mathbf{u}_i^T \mathbf{v}_j = \delta_{i,j}. \tag{2}$$

We have in view of (1) with $\mathbf{x} = C\mathbf{y}$:

$$R(\mathbf{x}) = \frac{y^T \Lambda y}{y^2} = \frac{\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2}{y_1^2 + \ldots + y_n^2}. \tag{3}$$

From this representation, it is immediately evident that

$$\min_{\mathbf{x}} R(\mathbf{x}) = \lambda_1 \quad \text{and} \quad \max_{\mathbf{x}} R(\mathbf{x}) = \lambda_n.$$

So we obtained representations of λ_1 and λ_n which are completely independent of any coordinates.

To obtain a similar representation for the rest of λ_j, we consider a minimization problem with restrictions. First of all

$$\min_{\mathbf{u}_1^T \mathbf{x}=0} R(\mathbf{x}) = \min_{y_1=0} R(\mathbf{x}) = \min \frac{\lambda_2 y_2^2 + \ldots + \lambda_n y_n^2}{y_2^2 + \ldots + y_n^2} = \lambda_2. \tag{4}$$

This is not very useful, because the knowledge of $\mathbf{u}_1$ is required. So take *any vector* $\mathbf{a}$ and consider the restriction $\mathbf{a}^T\mathbf{x} = 0$. We will show that minimum with this restriction is between λ_1 and λ_2. That it is at least λ_1 is clear because λ_1 is the unrestricted minimum. To show that it is at most λ_2, let us choose a vector $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \neq 0$, which satisfies the restriction $\mathbf{a}^T\mathbf{x} = 0$. The restriction gives *one* homogeneous linear equation on two unknowns c_1, c_2, namely

$$\mathbf{a}^T\mathbf{x} = c_1\mathbf{a}^T\mathbf{v}_1 + c_2\mathbf{a}^T\mathbf{v}_2 = 0$$

therefore such a non-zero solution (c_1, c_2) exists. For this vector $\mathbf{x}$,

$$y_1 = \mathbf{u}_1^T\mathbf{x} = c_1\mathbf{u}_1^T\mathbf{v}_1 + c_2\mathbf{u}_1^T\mathbf{v}_1 = c_1,$$

where we used (2), and similarly $y_1 = c_2$, and $y_k = 0$ for $k \geq 3$. So

$$R(\mathbf{x}) = \frac{\lambda_1 c_1^2 + \lambda_2 c_2^2}{c_1^2 + c_2^2} \leq \lambda_2,$$

and $\mathbf{x}$ satisfies the restriction. So the restricted minimum is between λ_1 and λ_2. Combined with (4) this can be stated as:

$$\max_{\mathbf{a}} \min_{\mathbf{a}^T\mathbf{x}=0} R(\mathbf{x}) = \lambda_2.$$

This gives a formula for λ_2 as a solution of a *maximin problem.* Completely similar reasoning gives the following:

Maximin Principle.

$$\lambda_k = \max_{\mathbf{a}_1,\ldots,\mathbf{a}_{k-1}} \left(\min_{\mathbf{a}_1^T\mathbf{x}=0,\ldots,\mathbf{a}_{k-1}^T\mathbf{x}=0} R(\mathbf{x}) \right).$$

Proof. When $\mathbf{a}_1 = \mathbf{u}_1, \ldots, \mathbf{a}_{k-1} = \mathbf{u}_{k-1}$ we have

$$\min_{\mathbf{u}_1^T\mathbf{x}=0,\ldots,\mathbf{u}_{k-1}^T\mathbf{x}=0} R(\mathbf{x}) = \min \frac{\lambda_k y_k^2 + \ldots + \lambda_n y_n^2}{y_k^2 + \ldots + y_n^2} = \lambda_1.$$

Now consider any $k-1$ restrictions

$$\mathbf{a}_1^T\mathbf{x} = \ldots \mathbf{a}_{k-1}^T\mathbf{x} = 0,$$

and choose a non-zero vector

$$\mathbf{x} = y_1\mathbf{v}_1 + \ldots + y_k\mathbf{v}_k$$

which satisfies the restriction. This is possible, becauses the restrictions are $k-1$ equations and we have k unknowns, so there is always a non-zero solution. For this vector

$$R(\mathbf{x}) = \frac{\lambda_1 y_1^2 + \ldots + \lambda_k y_k^2}{y_1^2 + \ldots + y_k^2} \le \lambda_k.$$

This completes the proof.

Similarly we could begin with maximizing $R(x)$ instead of minimizing. Then completely similar arguments give the

Minimax Principle

$$\lambda_k = \min_{\mathbf{a}_1,\ldots,\mathbf{a}_{n-k}} \left(\max_{\mathbf{a}_1^T\mathbf{x}=0,\ldots,\mathbf{a}_{n-k}^T\mathbf{x}=0} R(\mathbf{x}) \right).$$

Geometric interpretation.
Let us consider the case when $M = I$, then C is ortthogonal, and we have only one orthogonal basis $\mathbf{u}_j = \mathbf{v}_j, \quad 1 \le j \le n$, columns of C are the same

as rows of $C^T = C^{-1}$. and they are (ordinary) eigenvectors of A I recall that the set

$$E = \{\mathbf{x} : \mathbf{x}^T A\mathbf{x} = 1\}$$

is called an ellipsoid. Since the Rayleygh ratio is homogeneous,

$$\lambda_1 = \min_{\mathbf{x}} R(x) = \frac{1}{\max_{\mathbf{x}\in E} \frac{\|\mathbf{x}\|^2}{\mathbf{x}^T A\mathbf{x}}} = \frac{1}{\max_{\mathbf{x}\in E} \|\mathbf{x}\|^2},$$

so the length of the larger semi-axis is

$$\sqrt{\max_{\mathbf{x}\in E} \|\mathbf{x}\|^2} = \frac{1}{\sqrt{\lambda_1}}.$$

Now when we intersect our ellipsoid E with a hyperplane $\mathbf{a}^T\mathbf{x} = 0$, we obtain an ellipsoid (of dimension $n-1$) which has the largest semi-axis

$$\frac{1}{\sqrt{\max_{\mathbf{x}\in E:\mathbf{a}^T\mathbf{x}=0} \|\mathbf{x}\|^2}},$$

and this largest semi-axis is maximal when $\mathbf{a} = \mathbf{u}_1$, that is when the hyperplane is perpendicular to the largest semi-axis.

Applications.

1. Let us say that $A \geq B$ if $\mathbf{x}^T A\mathbf{x} \geq \mathbf{x}^T B\mathbf{x}$ for all $\mathbf{x} \neq 0$. This is equivalent to saying that $A - B$ is non-negative semi-definite. Indeed

$$\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T B\mathbf{x} = \mathbf{x}^T (A - B)\mathbf{x}.$$

Now consider four matrices A, M, A', M' all symmetric, and M, M' positive definite. Let λ_j be the generalized eigenvalues solving $\det(A - \lambda M) = 0$ and λ'_j the generalized eigenvalues solving $\det(A' - \lambda' M') = 0$.

Theorem 1. *If $A \geq A'$ and $M \leq M'$ then $\lambda_j \geq \lambda'_j$, for all j.*

Indeed, we have $R(\mathbf{x}) \geq R'(\mathbf{x})$ for the Rayleigh ratios, so all maxima and minima involving R are at least those involving R'.

This has a nice and useful physical interpretation. Recall the equation of small oscillation of a mechanical system. It is a second order differential equation

$$M\mathbf{y}'' + K\mathbf{y} = 0,$$

where M (mass) and K (stiffness) are symmetric matrices, and $M > 0$. So we obtain the following principle:

Increasing stiffness and/or decreasing mass of the system results in increasing all frequencies of proper oscillations.

2. Suppose that the matrix A' is obtained from a symmetric matrix A by deleting some columns and rows with the same numbers. For example, A' can be a NW submatrix of A. How are their eigenvalues related? Suppose that A is of size $n \times n$ and A' is of size $m \times m$, $m < n$. Let $\lambda'_1 \leq \lambda'_2 \leq \ldots \leq \lambda'_m$ be eigenvalues of A' and $\lambda_1 \leq \ldots \leq \lambda_n$ be eigenvalues of A.

Theorem 2. *We have*

$$\lambda_k \leq \lambda'_k \leq \lambda_{k+n-m}, \quad 1 \leq k \leq m. \tag{5}$$

In particular, when $n - m = 1$ we obtain

$$\lambda_1 \leq \lambda'_1 \leq \lambda_2 \leq \lambda'_2 \leq \ldots \leq \lambda'_{n-1} \leq \lambda_n.$$

This is called the *interlacing property*: the eigenvalues of the reduced matrix interlace with those of the original one.

Proof of Theorem 2. To prove $\lambda'_k > \lambda_k$ we use Maximin Principle. λ'_k can be written as maximum over $k-1$ restraints of minima $R(\mathbf{x})$ under these $k-1$ restrains and the restraints $x_{m+1} = \ldots = x_n = 0$. Removing the last $n - m$ restraints decreases the minimum of $R(\mathbf{x})$, and maximum over restraints of minima of $R(\mathbf{x})$ under $k - 1$ restraints is λ_k.

To prove $\lambda_k \leq \lambda_{k+n-m}$ we use Maximin Principle again. For $x \in \mathbf{R}^n$ we denote by $\mathbf{x}' \in \mathbf{R}^m$ the vector consisting of the first m coordinates of $\mathbf{x}$. Then

$$\lambda'_k = \max_{\text{restrictions}} \left(\min_{k-1 \text{ restrictions}} R'(x') \right)$$

But minimum of $R'(\mathbf{x})$ under $k - 1$ restrictions equals to the minimum of $R(\mathbf{x})$ with the same restrictions plus $n - m$ restrictions of the form $x_j = 0$ $m + 1 \leq j \leq n$. So the total number of restrictions is $k - 1 + n - m$. This does not exceed maximum of these minima over all possible $k - 1 + n - m$ restrictions and this is λ_{k+n-m}.

Instead of considering submatrices, we can impose arbitrary linear restrictions that is a restriction of the quadratic form on a subspace of dimension m. The result will be the same.

3. Suppose that a form $\mathbf{x}^T A'\mathbf{x}$ is obtained from $\mathbf{x}^T A\mathbf{x}$ by adding r squares of linearly independent linear forms:

$$\mathbf{x}^T A'\mathbf{x} = x^T A\mathbf{x} + \sum_{j=1}^{r} (L_j(\mathbf{x}))^2. \tag{6}$$

How are the eigenvalues of A and A' related? Let λ_j be eigenvalues of A and λ'_j eigenvalues of A', both sequences are ordered so that they increase.

Theorem 3. *If the forms A and A' are related as in (6) then we have*

$$\lambda'_k \geq \lambda_k, \quad 1 \leq k \leq n$$

and

$$\lambda'_k \leq \lambda_{k+r}, \quad 1 \leq k \leq n - r.$$

Proof. The first inequality is clear from either Maximin or from Minimax Principle, because we have $R' \geq R$, so whatever maxima or minima one takes, this inequality will be preserved.

The second inequality is derived from the Maximin Principle. λ_{k+r} is the maximum over restrictions of minima of $R(\mathbf{x})$ with $k + r - 1$ restrictions. It will decrease if we take maximum not over all possible restrictions but fix r of them to be $L_j(\mathbf{x}) = 0$. But then this is the same as the minimum of $R'(\mathbf{x})$ with $k - 1$ restrictions and this is λ'_k.