Inverses and transposes

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We begin with a theorem which gives an algorithm to determine whether a square matrix is invertible, and if it is, to find the inverse. First we prove an important characterization of invertible matrices which will eventually give us this algorithm. We will frequently use this notation $[A, B]$ for matrices A, B of the same height. This just means putting A and B together, first columns of A, then columns of B, to form a new matrix.

Theorem 1. *A square matrix A is invertible if and only if for every column $\mathbf{b}$ the equations $A\mathbf{x} = \mathbf{b}$ has a unique solution.*

Proof. If A is invertible, then we can solve the equation $A\mathbf{x} = \mathbf{b}$ by $\mathbf{x} = A^{-1}\mathbf{b}$ and this works for every $\mathbf{b}$. Let us prove the converse statement.

If for every $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ is solvable, solve it for $\mathbf{b} = e_1, e_2, \ldots, e_n$. Here e_j are the columns of the unit matrix. We obtain solutions $A\mathbf{x}_j = e_j$ for each j. Now put these solutions together: $B = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$. Then by the rule of multiplication of matrices

$$AB = A[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = [A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n] = [e_1, e_2, \ldots, e_n] = I.$$

Therefore B is the right inverse.

To show the existence of the left inverse, we use the representation

$$LPA = U$$

from the previous lecture. Here P is a permutation matrix, L is lower triangular with 1's on the main diagonal, and U is upper triangular. Since all equations $A\mathbf{x} = \mathbf{b}$ have solutions U must have n pivots, so all elements on the main diagonal of U are not zeros. This means that we can continue row operations now adding multiples of rows of U to *higher rows* to make the matrix U diagonal, with all diagonal elements non-zero. These row operations are equivalent to multiplication of U from the left on some *upper-triangular matrices* $L_{i,j}(\ell)$ with $i < j$. We will have

$$U_1 LPA = D,$$

where U_1 is upper triangular, L is lower triangular, both with 1's on the main diagonal, and D diagonal with non-zero entries on the main diagonal. So D is invertible, and $D^{-1}U_1LPA = I$, that is $D^{-1}U_1LP$ is the left inverse of A. As we have seen before, it must coincide with the right inverse (Lecture 2, Theorem 1).

We can conclude this discussion of inverses by stating the following equivalent conditions for a square matrix:

0. A is invertible.

1. A has a left inverse.

2. A has a right inverse.

3. Equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

4. Equation $A\mathbf{x} = 0$ has only trivial solution.

6. REF for A has a pivot in every column (same as $r(A) = n$).

Square matrices which *do not* satisfy these conditions are called *singular*. Thus

$$\text{invertible} = \text{non-singular}$$

and

$$\text{singular} = \text{non-invertible}.$$

Corollary. *A square upper- or lower- triangular matrix is invertible if and only if all entries on the main diagonal are different from* 0.

The proof of Theorem 1 also gives the following algorithm.

Step 1. Consider the $n \times 2n$ matrix $[A, I]$, and bring it to the row echelon form. If each of the first n columns of REF contains a pivot, then A is invertible, otherwise it is not.

Step 2. Continue row operations, adding multiples of lower rows to higher rows until the left half of the matrix becomes diagonal.

Step 3. Divide each row on its pivot, so that the left half of the matrix becomes the unit matrix, so that the whole $n \times 2n$ matrix becomes $[I, B]$. This B is the inverse, $B = A^{-1}$.

Example. Find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Begin with

$$[A, I] = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

and bring it to the REF:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

At this point we know that A is invertible, since there are 3 pivots.

Now do row Operation 1 to make the left half of the matrix diagonal, that is subtract the 3-d row from the 2-nd:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse matrix is the right half:

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Verify your answer by multiplying AA^{-1} or $A^{-1}A$.

For the future use, we will need the uniqueness of $A = LDU$ factorization of square matrices for which no exchange is required.

To prove this uniqueness suppose that we have two such factorizations:

$$A = L_1 D_1 U_1 = L_2 D_2 U_2,$$

where L_j are lower triangular, with 1's on the main diagonal, D_j are diagonal, and U_j are upper triangular, with 1's on the main diagonal. All matrices are square of the same size.

Then since all involved matrices are invertible, we have

$$L_2^{-1} L_1 D_1 = D_2 U_2 U_1^{-1}.$$

The LHS is lower triangular and the RHS is upper triangular, so both must be diagonal. Furthermore, on the main diagonal of the LHS stand the diagonal elements of D_1, and on the main diagonal of RHS stand diagonal entries of D_2. Therefore $D_1 = D_2$. Now since RHS is diagonal it is equal to D_1, and so $L_2 L_1^{-1} = I$ since D_1 is invertible. So $L_1 = L_2$. Similarly $U_1 = U_2$.

Transpose of a matrix. Another important operation on matrices is *transposition.* Transposition is a "reflection" of a matrix in its main diagonal: rows become columns and vice versa, preserving their order. Formally if $A = (a_{i,j})$, then the transpose $A^T = B = (b_{i,j})$, where $b_{i,j} = a_{j,i}$. So transposition of a $m \times n$ matrix is an $n \times m$ matrix. Transposition commutes with linear combinations:

$$(c_1 A + c_2 B)^T = c_1 A^T + c_2 B^T.$$

Doing transposition twice evidently returns the same matrix:

$$(A^T)^T = A.$$

For the transposition of the product we have

$$(AB)^T = B^T A^T.$$

Notice the inverse order! A formal proof of this rule is the following. Let $AB = C = (c_{i,j})$. Then by the rule of multiplication

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

So the elements of C^T are

$$c_{i,j}^T = c_{j,i} = \sum_k a_{j,k} b_{k,i} = \sum_k b_{i,k}^T a_{k,j}^T,$$

and these are the elements of $B^T A^T$. I leave the rule for the inverse as an exercise:

$$(A^T)^{-1} = (A^{-1})^T.$$

The precise statement: if A is invertible then also A^T is invertible, and the inverse of A^T is given by this formula.

Here is another exercise: for every permutation matrix P we have

$$P^{-1} = P^T.$$

So permutation matrices give an example of an important class of matrices which satisfy

$$A^{-1} = A^T.$$

Such matrices are called *orthogonal* and they will be studied later.

Another important class is *symmetric matrices* characterized by the property

$$A^T = A.$$

There are also *skew-symmetric matrices*: they are defined by the property

$$A^T = -A.$$

Show that all entries on the main diagonal of a skew-symmetric matrix are zeros.

Exercise. Is the product of orthogonal matrices always orthogonal? Is the product of symmetric matrices always symmetric? What about sums of these types of matrices?

One application of transposition is the convenience of writing and printing. Instead of writing a column vector, we can write $(x_1, x_2, x_3)^T$, which saves a lot of paper.