Four subspaces associated to a matrix

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In this section we fix two positive integers m, n and consider a matrix $A \in \mathrm{Mat}(m \times n)$. We denote by $\mathbf{R}^n$ the space of columns of height n and by $\mathbf{R}^{n*}$ the space of rows of length n.

To the matrix A we associate 4 subspaces:

1. The row space of A, denoted by $R(A)$. It is the span of the rows of A, and it is a subspace of the space of all rows $\mathbf{R}^{n*}$.

2. The nullspace of A, denoted by $N(A)$. It is the subspace of $\mathbf{R}^n$ which consists of solutions of the homogeneous linear system with coefficient matrix A:

$$N(A) = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

3. The column space of A, denoted by $C(A)$. It is the span of the columns of A and it is a subspace of $\mathbf{R}^m$.

4. The left nullspace of A defined by

$$N(A^T) := \{x \in \mathbf{R}^{m*} : \mathbf{x}^T A = \mathbf{0}\}.$$

Our goal in this section is to find a basis in each of these spaces, and to determine their dimensions.

To do this we perform row operations on A, and obtain its row echelon form U:

$$LPA = U.$$

To obtain a basis in $R(A)$ and $N(A)$ we notice that row operations do change neither $R(A)$ nor $N(A)$: when we perform an operation, the rows of the new matrix are linear combinations of the rows of the old matrix and vise versa, since the row operations are invertible. Therefore it is sufficient to find bases in $R(U)$ and $N(U)$.

A basis in $R(U) = R(A)$ consists of all non-zero rows of U. Indeed, they evidently span $R(U)$. To show that they are linearly independent, we can continue row operations (they do not change the row space) to make all entries *above* the pivot are equal to zero. Then it becomes evident that no non-trivial linear combination of the rows can be zero, since there is a single non-zero entry in each column where a pivot stands. So

A basis in $R(A)$ consists of non-zero rows of the REF, and

$$\dim R(A) = r(A),$$

where $r(A)$ is the rank of the matrix.

By the way we just proved that the rank is well defined, that is it depends only on A, not of the choice of the REF (REF is not unique!).

For $N(A)$ we do the following. Let $t_1, t_2, \ldots, t_{n-r}$ be the free variables. Their number is indeed $n - r$ since there are $r = r(A)$ pivots, one in each non-zero row of the REF. Assign to one free variable the value 1 and zero to the rest. Then we obtain $n - r$ solutions of $A\mathbf{x} = \mathbf{0}$, let us call them $\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}$. They form a basis of $N(A)$. Indeed, they span the set of all solutions (see Lecture 1), and they are linearly independent: for the matrix $[v_1, \ldots, \mathbf{v}_{n-r}]$ contains a unit submatrix of size $n - r$. Thus we have the following recipe:

A basis of $N(A)$ is obtained by assigning to each free variable the value 1 and assigning zero to the rest of them. The solutions of $A\mathbf{x} = \mathbf{0}$ thus obtained form a basis.

$$\dim N(A) = n - r(A).$$

This is called sometimes the "Rank and Nullity Theorem".

Now we consider the column space $C(A)$. Of course the columns of A are scrambled by the row operations, so their span does not remain the same. However there is another preserved feature: *linear dependencies between columns do not change when we perform row operations.* This fact seems evident, but here is a formal proof. Let $\mathbf{a}_1, \ldots, \mathbf{a}_k$ be some columns of

A, and suppose that they are linearly dependent

$$c_1\mathbf{a}_1 + \ldots + c_k\mathbf{a}_k = 0. \tag{1}$$

Performing row operations on these columns is equivalent to multiplying all of them on the same non-singular matrix (of type L or P in our standard notation). Multiplying (1) by a non-singular matrix X gives

$$c_1X\mathbf{a}_1 + \ldots + c_kX\mathbf{a}_k = \mathbf{0},$$

so linear dependencies are preserved.

This implies that if some columns of A with numbers $n_1, \ldots, n_k$ form a basis of the column space $C(A)$, then the columns of U with the *same numbers* form a basis of $C(U)$. So one only has to determine the position of these columns. And it is clear that the columns which contain th pivots form a basis in $C(U)$. Indeed, if we disregard all other columns of U we will obtain an upper-triangular matrix with non-zero elements (pivots!) on the main diagonal. So we obtain this rule:

To find a basis of $C(A)$, determine the positions of columns of U which contain pivots, and take the columns at the same positions in A. So

$$\dim C(A) = r(A)$$

(the number of pivots).

Finally we consider $N(A^T)$. The dimension is clear, just apply what we know about $N(A)$ to the transposed matrix. So

$$\dim N(A^T) = m - r.$$

To find a basis, one can apply the method described for $N(A)$ to the transposed matrix. This requires finding a REF for A^T, so it is time consuming. A simpler method is the following. Consider the equation

$$LPA = U$$

obtained in the process of finding the REF. The row echelon form U has $m - r$ zero rows. These zero rows are obtained by multiplying the last $m - r$ rows of LP by A. So each of these rows of LP is a solution of $\mathbf{x}^A = \mathbf{0}$. Next, they are linearly independent (all rows of the non-singular matrix LP

are linearly independent!) And finally we have the correct number $m-r$ of these rows, the number equal to $\dim N(A^T)$. Therefore these rows make a basis of $N(A^T)$. Thus

To find a basis of $N(A^T)$, bring A to the REF U and take the last $m-r$ rows of LP from $LPA = U$, and transpose them as columns.

Example 1.

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 2 & -1 \end{pmatrix}$$

The row echelon form is obtained subtracting the first row from the second, and subtracting twice the first row from the third. Then subtract second row from the third. This corresponds to multiplication of A from the left by

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad L = L_2 L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

and the row echelon form is

$$U = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we have

$$LA = U.$$

The rank is $r(A) = 2$ (=the number of non-zero rows in U). A basis of $R(A)$ is

$$(1, 2, 0, -1), \quad (0, -3, 2, 1).$$

To obtain a basis of $N(A)$, we notice that the free variables are x_3 and x_4. Assigning to them the values $(0, 1)$ and $(1, 0)$, we obtain the basis

$$(1/3, 1/3, 0, 1)^T, \quad (-4/3, 2/3, 1, 0)^T$$

(these are column vectors). To obtain a basis of $C(A)$ we notice that the columns in U that contain pivots are 1 and 2. So the basis is formed by columns 1 and 2 of A:

$$(1, 1, 2)^T, \quad (2, -1, 1)^T.$$

Finally, a basis of $N(A^T)$ is formed by the column which is transposed to the last row of L:

$$(-1, -1, 1)^T$$

.

Example 2. Find a basis in the plane

$$x + y + 2z = 0.$$

This plane is the null space of the matrix $A = (1, 1, 2)$. It is already in the row echelon form, and the free variables are y and z. So our procedure gives the basis

$$(-1, 1, 0)^T, \quad (-1/2, 0, 1)^T.$$