Midterm exam. Solutions and comments

1. a) $|z| < \pi/2$ (because $\pi/2$ is the singular point closest to the origin).

b) $|z| < 3^{-1/2}$ (by Hadamard's formula)

c) $|z| < e$. If one uses Hadamard's formula, one needs to know Stirling's formula

$$n! = n^n e^{-n}\sqrt{2\pi n + o(n)}.$$

The elementary approach is to use the ratio test.

$$|a_n/a_{n+1}| = |z|^{-1}(1 + 1/n)^n \to e/|z|,$$

so the series converges for $|z| < e$ and diverges for $|z| > e$.

d) This is not a power series, but a simple change of variable $w = e^z$ reduce it to a power series. Using this or arguing directly we conclude that the region of convergence is the right half-plane.

e) The RHS has a pole at the origin which is the closest singularity to the point 1, so the series is convergent for $|z - 1| < 1$.

2. a) essential singularity at $z = 1$. In d) and e), essential singularity at 0.

Most of you had difficulties with b) and c).

In b), $1 - \cos z$ has a *double* zero at the origin. The reciprocal has double poles at $2\pi n$, but the double pole at 0 is canceled by z^3. So the answer is: removable singularity at 0 and double poles at $2\pi n$ for all integers $n \neq 0$.

In c), $\cot \pi z$ has simple poles at the integers. and $2/(1 - z^2)$ has simple poles at ± 1. So these points ± 1 need additional investigation: The poles might cancel when we subtract two functions. For example $\cot z - 1/z$ has a removable singularity at the origin. In our case they do not cancel, because

$$\frac{2}{1 - z^2} = \frac{1}{1 - z} + \frac{1}{1 + z}$$

has residues 1 at both poles, while $\cot \pi z$ has residues $1/\pi$. To see this, we write

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} = \frac{1}{\pi z} + \dots$$

near zero. So the residue at 0 is $1/\pi$. Then it is the same at all other integers by periodicity. Thus the answer is: simple poles at all integers.

3. Make a substitution $w = e^{iz}$. Then

$$\frac{w - w^{-1}}{i(w + w^{-1})} = 2i.$$

This is a quadratic equation whose solutions are

$$w = \pm i\sqrt{3}.$$

So

$$z = -iw = i\mathrm{Log}\sqrt{3} \pm \frac{\pi}{2} + 2\pi k.$$

4. Parametrize the circle by $z = 1 + e^{it}$. Then the integral is equal to

$$\int_0^{2\pi} (1 + e^{it})(1 + e^{-it}) i e^{it} dt = i \int_0^{2\pi} (2e^{it} + e^{2it} + 1) dt = 2\pi i,$$

because the integrals of the exponentials from 0 to 2π are equal to zero. (Using sines, cosines and square roots was a bad idea: calculations with complex exponents are almost always *simpler* that trig calculations).

5. In a) some interpreted the "circle" literally, and others interpreted it as a "disc". The answer depends on the interpretation, and I gave full credit for a complete explanation in both cases.

"Does there exist a fractional-linear transformation which sends the unit *circle* to the right half-plane, 1/2 to 1 and 2 to -1?"

No. Because fractional-linear transformations send circles to circles, and a half-plane is not a circle. (References to 1-to-1 or continuity are not enough: there are 1-to-1 maps of a circle to a half-plane, and there are continuous maps of a circle to a right half-plane. It is true that there is no 1-to-1 *and* continuous map like this, but this is a deep result).

"Does there exist a fractional-linear transformation which sends the unit *disc* to the right half-plane, 1/2 to 1 and 2 to -1?"

Yes. $f(z) = 3(1 - z)/(1 + z)$.

b) No. Because 1/2 and 2 are symmetric with respect to the unit circle while 1 and 3 are not symmetric with respect to the imaginary axis. Or because 1/2 and 2 are in different components of the complement of the unit disc, while 1 and 3 are in the same component of the complement of the imaginary axis.