MATH 530

Practice Exam 1.

SEPTEMBER, 2007

Name:

There will be only 5 problems on the exam, each worth 10 points, some with items a), b), c), d), e), where each item is worth 2 points.

1. Determine the radii of convergence of the following series:
a) $\cot z = \sum_{n=0}^{\infty} a_n (z-1)^n$, b) $\sum_{n=0}^{\infty} n^{-n} z^n$, c) $\sum_{n=0}^{\infty} 2^n z^{n^2}$,
d) $\sum_{n=0}^{\infty} (n^n/n!) z^{n!}$, e) $z \cot z = \sum_{n=0}^{\infty} a_n (z-1)^n$.

2. Find an analytic function in the complex plane, whose real part is

$$e^{-x}(x \cos y + y \sin y),$$

where $z = x + iy$.

3. True or false: *if f is an analytic function in a region D, and $|f'(z)| \leq 1$ for all $z \in D$, then $|f(z_1) - f(z_2)| \leq |z_1 - z_2|$ for all z_1, z_2 in D*? Prove it, if true, or give a counterexample, if false.

4. Suppose that f is meromorphic in the unit disc $|z| < 1$, and has only one simple pole $z_0 \neq 0$ there. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of f at 0. Prove that

$$z_0 = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}.$$

5. Find and classify all isolated singularities of the following functions in $\overline{\mathbf{C}}$. (Singularities at ∞ should be considered too, if they are isolated. For poles, tell their multiplicities).

$$\frac{1}{e^z - 1} - \frac{1}{\sin z}, \quad \frac{\sin z}{z}, \quad \frac{z}{1 - \cos z}, \quad \sin \tan z.$$

6. Evaluate the integral

$$\int_{|z|=15} \frac{z^3 dz}{z^5 - z + 1}.$$

7. Find all solutions of the equation

$$\sin z = 5i$$

and sketch them in the complex plane.

8. For functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in the unit disc $|z| < 1$, prove
a) that f is even, if and only if $a_n = 0$ for all odd integers n.
b) if $f(x)$ is real for all $x \in (-1, 1)$, then $f(\overline{z}) = \overline{f(z)}$, for all z in the unit disc.

9. Suppose that f is an analytic function in the whole complex plane, which satisfies $f(z+1) \equiv f(z)$, and $f(z+i) \equiv f(z)$. Prove that this f is constant.

10. Let u be a non-constant harmonic function in the whole complex plane. Prove that the set $\{z : u(z) = 0\}$ is unbounded.

11. Which of the following interpolation problems are solvable for analytic functions in $|z| < 2$? Here $n = 1, 2, 3, \ldots$

a) $f(1/n) = (-1)^n$.

b) $f(1/n) = (-1)^n/n$.

c) $f(1/n) = (-1)^n n^{-2}$.

d) $f(1/n) = n/(n+1)$.

12. Prove that any two disjoint circles on the Riemann sphere can be mapped onto concentric circles by a fractional-linear transformation.