

Second Practice Exam

There will be only 5 problems like this on the exam.

1. Evaluate the integral

$$\int_0^\infty e^{-x^2} \cos 2x \, dx.$$

Hint: integrate $\exp(-z^2)$ over a rectangle, whose base is $[-R, R]$, and height is 1.

- 2.. Evaluate the integral

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx.$$

3. Find the partial fraction expansion of

$$\frac{1}{e^z - 1}.$$

Hints: first, find poles and residues, and guess the expansion on this basis. Then find the sum of your series of partial fractions, using residue theory, to prove that you actually obtained an expansion of the given function.

4. Find the sum of the series

$$\sum_{n=1}^\infty \frac{(-1)^n}{n^2}.$$

5. How many non-real solutions does this equation have?

$$\tan \pi z = az, \quad \text{where } a \in \mathbf{R}.$$

The answer depends on a , of course. Be sure to consider all possible cases for various values of a .

6. Prove that for every complex a , and for every integer $n \geq 2$ the equation

$$1 + z + az^n = 0$$

has at least one solution in the disc $|z| \leq 2$. Hint: Besides Rouché's theorem, you can use Vieta's formulas!

7. Solve the Dirichlet problem for the following region

$$D = \{z : \operatorname{Re} z > 0, |z - 2| > 1\},$$

with the boundary conditions: 0 on the vertical line, and 1 on the circle. Hint: Solving the Dirichlet problem for a round annulus $\{1 < |z| < R\}$, with boundary conditions constant on each circle, is easy!

8. Let D be the region which remains, when we remove from the unit disc the segments $\{z = re^{i\theta} : 1/2 < 1/2 \leq r < 1, \theta = 2\pi k/5\}$, where $k = 0, 1, \dots, 4$. Find a conformal map of this region D onto the unit disc.

9. Prove that all fractional-linear transformations, which satisfy

$$f \circ f \circ f \circ f \circ f = \operatorname{id},$$

are elliptic.

10. Let f be an analytic function, bounded on every vertical line. Put

$$M(x) = \sup\{|f(x + iy)| : -\infty < y < \infty\}.$$

Prove that the function $M(x)$ is *convex*, that is $M(ta + (1 - t)b) \leq tM(a) + (1 - t)M(b)$, for all real a, b , and $t \in [0, 1]$.