530 Final and Qual Preparation problems

There will be 8 problems in each exam, final and qual. Starred problems in this list are a bit harder than those in the exams. I do not include here the standard questions like evaluating an integral or counting roots of a polynomial in a disc, but they may be on the exams. You can make such questions yourself by changing the parameters in the problems in the book, or looking at the old quals on the web.

1. Let f be a rational function, $f(\infty) \neq \infty$. Prove that the equation $f(z) = z$ has at least one solution which satisfies $|f'(z)| > 1$ or $f'(z) = 1$.

Hint: look at the residues of $1/(f(z) - z)$.

2. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series with non-negative coefficients, with radius of convergence 1. Prove that the point 1 is singular (that is f cannot be analytic in any region containing the unit disc and the point 1).

Hint: re-expand the series about the point 1/2.

3. Consider the functional equation

$$f'(z) = f(kz), \quad f(0) = 1. \tag{1}$$

Prove the following:

a) If $|k| \leq 1$ then there is a unique function analytic at 0 that satisfies (1) and this function is entire.

b) If $|k| > 1$ then there is no analytic function at 0 that satisfies (1).

4. Prove the following generalization of Casorati–Sokhotski–Weierstrass theorem:

Let f be a meromorphic function in a region $\{z : 0 < |z - a| < r\}$. Then, either the limit

$$\lim_{z \to a} f(z)$$

exists, finite or infinite, or

$$\forall a \in \overline{\mathbf{C}} \quad \exists (\text{a sequence } z_n \to 0), \quad \text{such that} \quad f(z_n) \to a.$$

5. Let f and g be two analytic functions in the unit disc $\mathbf{U}$ such that f is injective, and

$$g(\mathbf{U}) \subset f(\mathbf{U}), \quad f(0) = g(0).$$

Prove that $|g'(0)| \leq |f'(0)|$. Describe when exactly equality is possible.

6. In the second statement of the Schwarz lemma, is the condition $f(0) = 0$ really needed? Let f be an analytic function mapping the unit disc into itself. Is it always true that $|f'(0)| \leq 1$?

If not, give an example. If yes, prove it.

7. Does there exist a surjective holomorphic map from $\mathbf{C}$ to the unit disc? Does there exist a surjective holomorphic map from the unit disc to $\mathbf{C}$?

8*. Does there exist a surjective holomorphic map from the unit disc to $\mathbf{C}$ whose derivative is never zero? (I will not give this in an exam. But if you solve it without external help, please send me your solution).

9. Let f and g be two entire functions, and $h = f \circ g$. Suppose that $g'(z) \neq 0$ for all $z \in \mathbf{C}$. Prove that the following statements are equivalent:

a) h has a zero of multiplicity m at z, and

b) f has a zero of multiplicity m at $g(z)$.

10. Let f be a function analytic in a neighborhood of 0. Suppose that

$$\sum_{n=0}^{\infty} f^{(n)}(z)$$

is convergent. Prove that there exists an entire function F such that $F(z) = f(z)$ in a neighborhood of 0, and that

$$\sum_{n=0}^{\infty} F^{(n)}$$

is convergent uniformly on compact subsets of $\mathbf{C}$.

11. Find a bounded harmonic function u in the half-disc $\{z : |z| < 1, \ \Re z > 0\}$ such that

$$\lim_{z \to \zeta} u(z) = \begin{cases} 0, & |\zeta| = 1, \ \Re\zeta > 0, \\ 1. & -1 < \zeta < 1. \end{cases}$$

12. Prove that

$$\frac{(a-b)(d-c)}{(a-c)(d-b)}$$

is a real number if and only if a, b, c, d belong to some circle or straight line.

13*. Circles in this problem mean Euclidean circles in the plane. Suppose that two circles C_1 and C_2 are given, they are disjoint and C_1 is inside C_2. Now let us draw circles $c_0, c_1, \ldots$ by the following rules. All of them are touching C_2 from inside and C_2 from outside. Choose c_0 arbitrary. Then let c_1 touch c_2, c_2 touch c_3 and so on. If $c_n = c_0$ for some n, we call the smallest such n the outcome of the game. If this never happens, we say that outcome is ∞.

Prove that the outcome does not depend on the choice of the initial circle c_0, but depends only on C_1 and C_2.

Hint: this is a corollary of one of the homework problems.

14. Let f be an analytic function in the unit disc, $f(0) = 0$. Prove that

$$g(z) = \sum_{n=0}^{\infty} f(z^n)$$

is well defined in the unit disc (that is the series is uniformly convergent on every compact subset of the unit disc.)

15. Consider the region $D = \mathbf{C} \backslash (L_1 \cup L_2)$ where L_1 and L_2 are two rays, $L_1 = [1, +\infty)$ and $L_2 = (-\infty, -1]$.

a) Show that the expression

$$f(w) = -i\mathrm{Log}\,(iw + \sqrt{1 - w^2})$$

where Log and $\sqrt{}$ are the principal branches, defines an analytic function in D.

b) Show that

$$f'(w) = \frac{1}{\sqrt{1 - w^2}} \quad \text{principal branch.}$$

c) Show that $\sin f(w) = w$ in D.

d) Is it true that $f(\sin z) = z$ in the whole plane? If not, give an example of a point z where $f(\sin z) = z$ does not hold.