Midterm solutions

1. Integrate by the curve consisting of the vertical segment $[1-iM, 1+iM]$ and the left half-circle having this segment as diameter. The contribution from the half-circle tends to zero as $M \to \infty$. So the intergal is equal to $2\pi i$ times the sum of the residues. The poles are at $\pm i$ and they are simple. Computing the residues we obtain the answer: the integral equals $2\pi i \sin 1$.

2. Rouché's theorem gives one root in the unit circle and four roots in the circle $|z| < 2$. So the answer is 3.

3. Suppose that $g(z_1) = g(z_2)$, where $g(z) = z + f(z)$. As the region is convex, it contains the segment $S = [z_1, z_2]$. Then we can write

$$|z_1 - z_2| = |f(z_1) - f(z_2)| \leq \left| \int_{z_1}^{z_2} f'(\zeta) d\zeta \right| \leq \max_I |f'(\zeta)||z_1 - z_2|,$$

This is a contradiction because f' is a continuous function, and the segment S is compact, so $\max_I |f'(\zeta)| < 1$.

4. $a_{2n+1} = (-1)^n \sum_{k=0}^n 1/(2k+1)!, \quad n = 0, 1, 2, \ldots.$

5. $z = 1$ is removable, because Log has a zero at this point. So the function is analytic (more precisely, has an analytic continuation) in the plane minus the negative ray. So the radius of convergence is at least 2, the distance from the point 2 to the boundary of this region. It cannot be greater than 2 because the function does not have an analytic continuation to the point 0, for example, because it tends to infinity as $z \to 0$.

6. By the general formula proved in class, the integral equals

$$\mathrm{Re}\left((2\pi i \mathrm{res}_{\exp(\pi i/3)} + \pi i \mathrm{res}_{-1}) \frac{e^{iz}}{1+z^3} \right).$$

Computing the residues:

$$\mathrm{res}_{-1} \frac{e^{iz}}{1+z^3} = \frac{e^{-i}}{3(-1)^2} = e^{-i}/3,$$

and

$$\mathrm{res}_{\exp(\pi i/3)} = \frac{e^{ie^{\pi i/3}}}{3e^{2\pi i/3}} = \frac{1}{3} e^{-2\pi i/3} e^{-\sin(\pi/3)} (\cos\cos(\pi/3) + \sin\cos(\pi/3)).$$

Here we used the formula
$$\mathrm{res}_a \frac{f}{g} = \frac{f(a)}{g'(a)},$$
if a is a simple zero of g and $f(a) \neq 0$. After taking the real parts we obtain the answer:
$$\frac{\pi}{3}\left(\sin 1 + e^{-\sqrt{3}/2}(\sin(1/2) + \sqrt{3}\cos(1/2))\right).$$

7. a) essential singularity at 0,

b) removable singularity at 0 and simple poles at $\pi k, k \in \mathbf{Z}\backslash\{0\}$.

c) essential singularities at the points $1/(\pi/2 + \pi k),\ k \in \mathbf{Z}$. (Zero is not an isolated singularity. Other singularities accumulate to it.)

d) double poles at $2\pi k,\ k \in \mathbf{Z}$.