

When a meromorphic function that omits three values has a bounded type

Alexandre Eremenko, Aleksei Kulikov, Mikhail Sodin

April 6, 2026

Abstract

Suppose that a function F is meromorphic in the domain $\mathbb{H}(-m) = \{z: \operatorname{Im} z > -m(\operatorname{Re} z)\}$, where m is an even, positive, and continuous function that does not increase on $\mathbb{R}_{\geq 0}$ type X (return,,,,i,,i,,i) there three distinct values. Then F is of bounded type in the upper half-plane (i.e., is represented there as a quotient of two bounded analytic functions), provided that the logarithmic integral of the function m is convergent. On the other hand, if the logarithmic integral of m diverges, there exists a function F meromorphic in $\mathbb{H}(-m)$, that omits there three distinct values, and which is of unbounded type in the upper half-plane.

This result is motivated by a century old question originating with Rolf Nevanlinna, which remains unsettled.

1 Introduction

We use the following notation:

- $m: \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is an even continuous function that does not increase on $\mathbb{R}_{\geq 0}$.
- \mathbb{H} denotes the upper half-plane;
- $\mathbb{H}(\pm m) = \{z = x + iy: y > \pm m(x), x \in \mathbb{R}\}$.
- For $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$ and $x^- = (-x)^+$.
- $A \lesssim B$ means that $A \leq CB$ with a positive constant C , $A \gtrsim B$ means that $B \lesssim A$, and $A \simeq B$ means that $A \lesssim B$ and $A \gtrsim B$ hold simultaneously.

A meromorphic function F is said to be of *bounded type* (belongs to the *Nevanlinna class*) in a domain $G \subset \mathbb{C}$ if it can be represented as $F = F_1/F_2$, where F_1, F_2 are bounded

analytic functions in G . Otherwise, F is said to be of *unbounded type* in G . In this note we prove the following theorem.

Theorem 1.

(a) Assume that $\int_1^\infty \frac{\log^- m(t)}{t^2} dt < \infty$. Then any meromorphic function F in $\mathbb{H}(-m)$ that omits three distinct values in $\mathbb{H}(-m)$ is of bounded type in \mathbb{H} .

(b) Assume that $\int_1^\infty \frac{\log^- m(t)}{t^2} dt = \infty$. Then there exists a function F meromorphic in $\mathbb{H}(-m)$ that omits three distinct values in $\mathbb{H}(-m)$ and is of unbounded type in \mathbb{H} .

This result arose as a by-product of our attempt to (negatively) resolve the following question originating in Nevanlinna's work [?]:

Question 2. Let F be a meromorphic function in \mathbb{C} . Suppose that it omits in a half-plane three distinct values from the extended complex plane. Is it of bounded type in that half-plane?

It is well-known that the elliptic modular function [?, Chapter 7, Section 3.4] is analytic in the upper half-plane, omits there the values 0, 1 and ∞ , and is not of bounded type there (we will provide details while proving part (b) of Theorem ??). It is also well-known [?, Chapter VI, item 145] that if a meromorphic function in a simply connected region omits a set of positive logarithmic capacity, then it is of bounded type.

Acknowledgement

The work of Aleksei Kulikov was supported by the United States – Israel Binational Science Foundation BSF Grant 2020019, the Israel Science Foundation ISF Grant 1288/21, and by the VILLUM Centre of Excellence for the Mathematics of Quantum Theory (QMATH) with Grant No.10059. The work of Mikhail Sodin was supported by the Israel Science Foundation ISF grants 1288/21, 2319/25, and the United States – Israel Binational Science Foundation BSF grants 2020019, 2024142.

2 Functions of bounded type in the upper half-plane

2.1 Nevanlinna's work

The question goes back to Nevanlinna's work [?] on the distribution of values of meromorphic functions in a half-plane. Therein, Nevanlinna introduced the characteristics of functions meromorphic in the *closed* upper half-plane $\overline{\mathbb{H}}$. Let f be a function meromorphic in $\overline{\mathbb{H}}$ (that is, meromorphic in a domain $G_f \supset \overline{\mathbb{H}}$), and $\{w_n\}$ be the set of its poles in \mathbb{H} ,

counted with multiplicities. Then

$$\begin{aligned}
c(r; f) &= \sum_{1 < |w_n| \leq r} \sin(\arg w_n), \\
C(r; f) &= 2 \int_1^r c(t; f) \left(\frac{1}{t^2} + \frac{1}{r^2} \right) dt \\
&= 2 \sum_{1 < |w_n| < r} \left(\frac{1}{|w_n|} - \frac{|w_n|}{r^2} \right) \sin(\arg w_n), \\
A(r; f) &= \frac{1}{\pi} \int_1^r \left(\frac{1}{t^2} - \frac{1}{r^2} \right) [\log^+ |f(t)| + \log^+ |f(-t)|] dt, \\
B(r; f) &= \frac{2}{\pi r} \int_0^\pi \log^+ |f(re^{i\theta})| \sin \theta d\theta, \\
S(r; f) &= A(r; f) + B(r; f) + C(r; f).
\end{aligned}$$

We list the basic asymptotic properties of these characteristics for $r \rightarrow \infty$:

1. If D is any of the characteristics A , B , and C , then $D(r; f_1 f_2) \leq D(r; f_1) + D(r; f_2)$ and $D(r; f_1 + f_2) \leq D(r; f_1) + D(r; f_2) + \log 2$. In particular, $S(r; f_1 f_2) \leq S(r; f_1) + S(r; f_2)$ and $S(r; f_1 + f_2) \leq S(r; f_1) + S(r; f_2) + 3 \log 2$.
2. For $a \in \mathbb{C}$, $S(r; 1/(f - a)) = S(r; f) + O(1)$ with an implicit constant depending on a .
3. There exists a non-decreasing function $S_o(r; f)$ such that $S(r; f) = S_o(r; f) + O(1)$.
4. $S(r; f)$ is bounded if and only if the function f is of bounded type in \mathbb{H} .

The first property is straightforward. The second one is an analogue of the first fundamental theorem. The proof of the second property [?, Chapter I, Theorem 5.1] is based on the Carleman integral formula. One way to prove the third property is to introduce the Ahlfors–Shimizu geometric form of the characteristics S :

$$S_o(r; f) = \frac{1}{\pi} \int_1^r \left(\frac{1}{t} - \frac{t}{r^2} \right) \left[\int_0^\pi \rho_f(te^{i\theta})^2 \sin \theta d\theta \right] t dt$$

where $\rho_f = |f'|/(1 + |f|^2)$ is the spherical derivative of f . The function $r \mapsto S_o(r; f)$ is non-decreasing, and a computation [?, Chapter I, Theorem 5.4] shows that

$$S(r; f) = S_o(r; f) + O(1), \quad r \rightarrow \infty.$$

In the fourth property, the direction

$$f \text{ is of a bounded type} \implies S(r; f) \text{ is bounded}$$

follows from the Nevanlinna factorization of the functions of bounded type in \mathbb{H} , since for each factor in the Nevanlinna factorization the characteristics S stays bounded. The opposite direction was proven in [?, Satz 1, p. 31]¹.

Nevanlinna showed [?, Satz II and Hilfssatz, p. 18] that *if F is a meromorphic function in \mathbb{C} of finite order of growth which omits in \mathbb{H} three distinct values, then $S(r; F) = O(1)$ for $r \rightarrow \infty$, and therefore, F is of bounded type in \mathbb{H} . That is, Question 2 has positive answer provided that F has finite order of growth. The proof followed the same strategy as his proof of the second fundamental theorem and was based on estimates of the characteristics $A(r; F'/F)$ and $B(r; F'/F)$ of the logarithmic derivative.*

2.2 Ostrovskii's refinement

At the beginning of the 1960s, Ostrovskii improved Nevanlinna's result and showed that the assumption that the meromorphic function F has finite order can be replaced by the weaker condition

$$\int_1^\infty \frac{\log^+ T(r; F)}{r^2} dr < \infty, \quad (1)$$

where $T(r; F)$ is the usual Nevanlinna characteristics of meromorphic functions in the complex plane. In particular, if the function F is entire, then an equivalent condition is

$$\int_1^\infty \frac{\log^+ \log^+ M(r; F)}{r^2} dr < \infty,$$

where $M(r; F) = \max_{|z| \leq r} |F(z)|$. His proof was also based on a careful estimate of $A(r; F'/F) + B(r; F'/F)$ [?, Chapter III, Theorems 3.1 and 3.3] (the more difficult part here

¹ Here is a brief sketch of the proof. Boundedness of the characteristics $C(r; f)$ implies that the poles of f satisfy the Blaschke condition in \mathbb{H} . By property 2 of $S(r; f)$, it also implies boundedness of $C(r; 1/f)$, i.e., the zeroes of f also satisfy the Blaschke condition. Therefore, we can separate from f the Blaschke products \mathcal{B}_0 and \mathcal{B}_∞ corresponding to its zeroes and poles. Similarly, boundedness of $A(r, f)$ and $A(r, 1/f)$ allows us to define the outer factor \mathcal{O}_f such that $|\mathcal{O}_f(x)| = |f(x)|$, $x \in \mathbb{R}$. Separating these factors from f , we arrive at a meromorphic function G in \mathbb{H} , having neither zeroes, nor poles in \mathbb{H} and satisfies $|G| = 1$ on \mathbb{R} . Let $h = \log |G|$. It is a harmonic function in \mathbb{H} that continuously vanishes on \mathbb{R} and satisfies

$$\int_0^\pi h^+(re^{i\theta}) \sin \theta d\theta = O(r), \quad r \rightarrow \infty.$$

By the Phragmén–Lindelöf principle, this yields that $h(z) = a \operatorname{Im} z$ with $a \in \mathbb{R}$. Thus, $G(z) = \Theta e^{iaz}$, where Θ is a unimodular constant. Then

$$f(z) = \Theta \frac{\mathcal{B}_0(z)}{\mathcal{B}_\infty(z)} \mathcal{O}_f(z) e^{iaz},$$

and therefore, f is of bounded type.

is the estimate of $A(r; F'/F)$). That is, *Question 2 has a positive answer for meromorphic functions satisfying condition (??)*.

Apparently, Nevanlinna hoped² to give a positive answer to Question 2 based on estimates of the logarithmic derivative F'/F . This hope evaporated after Gol'dberg [?] constructed an example of a meromorphic function F in \mathbb{C} with $S(r; F) \equiv 0$, while $A(r; F'/F)$ grows faster than any given function tending to ∞ as $r \rightarrow \infty$.

2.3 Edrei's theorem

Another related result is due to Edrei [?, Theorem 1]. It states that *if F is a meromorphic function in \mathbb{C} having only real a -, b -, and c -points (a, b, c are distinct values from the extended complex plane), then the order of growth of F does not exceed one*. A remarkable feature of this theorem is an absence of any priori assumption on the growth of F . Combining Edrei's theorem with the aforementioned theorem of Nevanlinna, we conclude that F has bounded type in the upper and lower half-planes. Entire functions possessing this property are called *entire functions of Cartwright class*.

3 Proof of Theorem ??

3.1 Preliminaries

First, we regularize the function m . We say that the function m is *tame* if, in addition to being even and non-increasing on $\mathbb{R}_{\geq 0}$, it is C^2 -smooth and satisfies the following conditions:

1. $m(x), |xm'(x)|, x^2|m''(x)| \rightarrow 0$ as $|x| \rightarrow \infty$;
2. it is constant on the intervals $|x| \leq 1$ and $2^k - 2^{k-3} \leq |x| \leq 2^k + 2^{k-2}$, $k \geq 3$.

Lemma 3. *Suppose that $m: \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is an even continuous function non-increasing on $\mathbb{R}_{\geq 0}$.*

(i) *If the logarithmic integral $\int_{\mathbb{R}} \frac{\log^- m(x)}{1+x^2} dx$ converges, then the function m has a tame minorant $m_* \leq m$ whose logarithmic integral also converges.*

(ii) *If the logarithmic integral $\int_{\mathbb{R}} \frac{\log^- m(x)}{1+x^2} dx$ diverges, then the function m has a tame majorant $m^* \geq m$ whose logarithmic integral also diverges.*

² He wrote in [?, p.18]: "...dass das Restglied $R(r)$ im allgemeinen von niedrigerer Grössenordnung als die Fundamentalgrösse $S(r, f)$ ist. Wir wollen bei dieser Gelegenheit auf eine allgemeine Untersuchung des Restgliedes nicht eingehen, sondern beschränken uns auf den einfachsten Fall, wo $f(x)$ in der ganzen endlichen Ebene eindeutig, meromorph und von endlicher Ordnung ist."

Proof of Lemma ??: First, we assume that the logarithmic integral converges. Set

$$m_*(x) = \begin{cases} m(8), & |x| \leq 1, \\ \min\{m(2^{k+3}), 2^{-k-3}\}, & 2^k - 2^{k-3} \leq |x| \leq 2^k + 2^{k-2}, k \geq 3. \end{cases}$$

We then smooth out the jumps of m_* , keeping m_* even and non-increasing on $\mathbb{R}_{\geq 0}$. Since $m_*(x) \leq \frac{1}{x}$, we clearly have $m(x) \rightarrow 0$. Since the size of the jump between the points $2^k + 2^{k-2}$ and $2^{k+1} - 2^{k-2}$ is $o(1)$ as $k \rightarrow \infty$, after the smoothing the function m_* will satisfy condition 1 in the definition of tameness. Furthermore, for $2^k \leq |x| \leq 2^{k+1}$, $m_*(x) \leq m(2^{k+3}) \leq m(x)$, so m_* is indeed a minorant of m . At last, by monotonicity of the functions m and m_* ,

$$\begin{aligned} \int_1^\infty \frac{\log^- m(x)}{x^2} dx < \infty &\implies \sum_k \frac{\log^- m(2^k)}{2^k} < \infty \\ &\implies \sum_k \frac{\log^- m_*(2^k)}{2^k} < \infty \implies \int_1^\infty \frac{\log^- m_*(x)}{x^2} dx < \infty, \end{aligned}$$

where in the second step we used that $\sum_k \frac{\log^-(2^{-k-3})}{2^k}$ clearly converges, so the extra 2^{-k-3} in the definition of $m_*(x)$ does not affect us.

The proof in the second case (the logarithmic integral diverges) is quite similar. First, we note that $m(x) = o(1)$ as $x \rightarrow \infty$, since the logarithmic integral is divergent and m does not increase. We set

$$m^*(x) = \begin{cases} m(0), & |x| \leq 1, \\ m(2^{k-3}), & 2^k - 2^{k-3} \leq |x| \leq 2^k + 2^{k-2}, k \geq 3. \end{cases}$$

Then we smooth out the jumps of m^* , keeping m^* even and non-increasing on $\mathbb{R}_{\geq 0}$, so that after the smoothing the function m^* will satisfy condition 1 in the definition of tameness. Furthermore, by construction, m^* is indeed a majorant of m , and by monotonicity of the functions m and m^* , the logarithmic integrals of m and m^* are equi-divergent:

$$\int^\infty \frac{\log^- m(x)}{x^2} dx = \infty \implies \int^\infty \frac{\log^- m^*(x)}{x^2} dx = \infty.$$

□

Lemma 4. *Suppose that the function m is tame, and let $w: \mathbb{H} \rightarrow \mathbb{H}(\pm m)$ be a biholomorphic map such that $w(iy)$ is purely imaginary and $w(iy) \rightarrow \infty$ as $y \rightarrow \infty$. Then the function w' extends continuously to the real axis, and $|w'(z)| \simeq 1$ everywhere in the closed half-plane $\overline{\mathbb{H}}$.*

Proof of Lemma ??: We will present the proof for the domain $\mathbb{H}(-m)$. For the domain $\mathbb{H}(m)$, the difference is purely notational.

The result follows from the classical Kellogg theorem (see, for instance, [?, Chapter II, Theorem 4.3]). It yields that if Ω is a bounded Jordan domain with a C^2 -smooth boundary, and $p: \mathbb{D} \rightarrow \Omega$ is a biholomorphic map, then p extends to a C^1 -diffeomorphism from $\overline{\mathbb{D}}$ to $\overline{\Omega}$ and p' does not vanish on $\partial\mathbb{D}$. To apply it here, we pre-compose our map w with the Cayley transform $C(\zeta) = i(1 + \zeta)/(1 - \zeta)$ and post-compose it with $J(w) = 1/(w + iA)$, where $A > m(0)$, letting $p = J \circ w \circ C$. The function p maps the unit disk onto the Jordan domain $J\mathbb{H}(-m)$. We claim that the boundary of $J\mathbb{H}(-m)$ is C^2 -smooth. This is evident everywhere except at the origin (J maps infinity to the origin). To check the smoothness at the origin, we write

$$h(x) = \frac{1}{x + i(A - m(x))},$$

and let

$$H(s) = \begin{cases} h(1/s), & s \neq 0 \\ 0, & s = 0. \end{cases}$$

Then $J\mathbb{H}(-m) = \{\zeta = \xi + i\eta: \xi > -H(\eta)\}$. So, we need to check that H is a C^2 -function on $[-1, 1]$. Since H is continuous on $[-1, 1]$, and C^2 on $[-1, 0) \cup (0, 1]$, it suffices to check that the limits of H' and H'' as $s \rightarrow \pm 0$ exist and coincide: $H'(+0) = H'(-0)$ and $H''(+0) = H''(-0)$. Let $\varphi(s) = 1 + is(A - m(1/s))$, $\varphi(0) = 1$. Then $H(s) = s/\varphi(s)$. We have $\varphi'(s) = i(A - m(1/s)) + im'(1/s)/s$ and $\varphi''(s) = -im''(1/s)/s^3$. By the tameness of m , we have $\varphi'(s) = iA + o(1)$ and $s\varphi''(s) = o(1)$, as $s \rightarrow 0$. Thus,

$$\begin{aligned} H'(s) &= \frac{1}{\varphi(s)} - \frac{s\varphi'(s)}{\varphi(s)^2} \\ &= 1 + o(1), \quad s \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} H''(s) &= -2 \frac{\varphi'(s)}{\varphi(s)^2} + 2 \frac{s\varphi'(s)^2}{\varphi(s)^3} - \frac{s\varphi''(s)}{\varphi(s)^2} \\ &= -2iA + o(1), \quad s \rightarrow 0, \end{aligned}$$

proving the claim. So, Kellogg's theorem can be applied.

Thus, p' extends to a continuous non-vanishing function in the closed unit disk. For $\zeta \rightarrow 1$ within \mathbb{D} , we have

$$p(\zeta) = b(\zeta - 1) + o(|\zeta - 1|), \quad b = p'(1) \neq 0.$$

Furthermore,

$$p'(\zeta) = J'(w(C(\zeta)))w'(C(\zeta))C'(\zeta), \quad C'(\zeta) = 2i/(1 - \zeta)^2,$$

and

$$J'(w) = -1/(w + iA)^2 = -J(w)^2.$$

Hence, for $\zeta \rightarrow 1$, $\zeta \in \mathbb{D}$,

$$\begin{aligned} J'(w(C(\zeta)))C'(\zeta) &= -p(\zeta)^2 C'(\zeta) \\ &= -(b(\zeta - 1) + o(|\zeta - 1|))^2 \cdot \frac{2i}{(1 - \zeta)^2} \rightarrow -2ib^2, \end{aligned}$$

and therefore,

$$\begin{aligned} \lim_{z \rightarrow \infty} w'(z) &= \lim_{\zeta \rightarrow 1} w'(C(\zeta)) \\ &= \lim_{\zeta \rightarrow 1} \frac{p'(\zeta)}{J'(w(C(\zeta)))C'(\zeta)} \\ &= \frac{b}{-2ib^2} = \frac{i}{2b}, \end{aligned}$$

proving more than what was stated in Lemma ???. □

3.2 Proof of Theorem 1(a)

Proving Theorem 1(a), we assume that the function m is tame. Otherwise, we replace m by its tame minorant m_* with convergent logarithmic integral $\int_{\mathbb{R}} \frac{\log^- m_*(x)}{1 + x^2} dx$, which exists by Lemma ??. Then $\mathbb{H}(-m_*) \subset \mathbb{H}(-m)$; so, if a meromorphic function F omits three distinct values in $\mathbb{H}(-m)$, a fortiori, it omits them in $\mathbb{H}(-m_*)$.

Consider the function $G = F \circ w$, where $w: \mathbb{H} \rightarrow \mathbb{H}(-m)$ is the Riemann map, as above. It is meromorphic in \mathbb{H} and omits three values therein. Hence, by Montel's theorem, the family of meromorphic functions $\{G \circ S: S \in \text{Aut}(\mathbb{H})\}$ is normal in \mathbb{H} . Then, following Lehto and Virtanen [?, Theorem 3], we get³

$$\rho_G(\zeta) \lesssim \frac{1}{\text{Im}(\zeta)}, \quad \zeta \in \mathbb{H}. \quad (2)$$

³ For the reader's convenience, we will reproduce their nice short argument. Consider the normal family \mathcal{M} of functions holomorphic in \mathbb{H} and omitting the values 0 and 1 therein. By Martý's lemma (a version of the Arzelá–Ascoli theorem), the spherical derivatives ρ_f of functions from \mathcal{M} are locally equibounded in \mathbb{H} . In particular, $\sup_{f \in \mathcal{M}} \rho_f(i) < \infty$. The family \mathcal{M} is invariant under the action of the group $\text{Aut}(\mathbb{H}) = SL_2(\mathbb{R})$. So, let

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad \alpha\delta - \beta\gamma = 1.$$

Then

$$\rho_{f \circ S}(i) = \rho_f(Si) |S'(i)| = \rho_f(S(i)) \frac{1}{\gamma^2 + \delta^2} = \rho_f(S(i)) \text{Im} S(i).$$

Thus, $\sup_{f \in \mathcal{M}} \rho_f(z) \text{Im} z = \sup_{f \in \mathcal{M}} \rho_f(i) < \infty$.

Whence, we get

$$\begin{aligned}
\rho_F(z) &= |(w^{-1})'(z)|\rho_G(w^{-1}(z)) \\
&\stackrel{(\ref{??})}{\lesssim} \frac{1}{\operatorname{Im}(w^{-1}(z))} \\
&= \frac{1}{\operatorname{dist}(w^{-1}(z), \partial\mathbb{H})} \\
&\simeq \frac{1}{\operatorname{dist}(z, \partial\mathbb{H}(-m))}, \quad z \in \mathbb{H}(m), \tag{3}
\end{aligned}$$

where we used Lemma ?? in the second and fourth steps. To estimate $\operatorname{dist}(z, \partial\mathbb{H}(-m))$, we use a simple geometric claim.

Claim 5. *Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 -function with bounded derivative $L = \|\varphi'\|_\infty < \infty$. Then, for every point (x, y) lying above the graph Γ_φ of φ , we have*

$$\operatorname{dist}((x, y), \Gamma_\varphi) \geq \frac{y - \varphi(x)}{\sqrt{1 + L^2}}.$$

Proof. Take any point $(t, \varphi(t)) \in \Gamma_\varphi$ and set

$$a = y - \varphi(x) > 0, \quad s = |t - x| \geq 0.$$

Then $y - \varphi(t) \geq (y - \varphi(x)) - L|t - x| = a - Ls$, and

$$|(x, y) - (t, \varphi(t))|^2 = (x - t)^2 + (y - \varphi(t))^2 \geq s^2 + (a - Ls)^2$$

whenever $a \geq Ls$, and the RHS is never less than $a^2/(1 + L^2)$. If $a < Ls$, then $s > a/L$, and

$$|(x, y) - (t, \varphi(t))| \geq |t - x| = s > \frac{a}{L} > \frac{a}{\sqrt{1 + L^2}}.$$

Thus, in both cases,

$$|(x, y) - (t, \varphi(t))| \geq \frac{y - \varphi(x)}{\sqrt{1 + L^2}}, \quad t \in \mathbb{R},$$

proving the claim. □

Combining (??) with the claim, we get

$$\rho_F(x + iy) \lesssim \frac{1}{y + m(x)}, \quad x + iy \in \mathbb{H}(-m). \tag{4}$$

Now, we are ready to see that the Nevanlinna characteristics $S(r, F)$ of F in the upper

half-plane remains bounded. For this, we use its Ahlfors–Shimizu version:

$$\begin{aligned}
S_o(r, F) &= \frac{1}{\pi} \int_1^r t \, dt \int_0^\pi \left(\frac{1}{t} - \frac{t}{r^2} \right) \rho_F(te^{i\theta})^2 \sin \theta \, d\theta \\
&\stackrel{(\text{??})}{\lesssim} \int_1^r dt \int_0^\pi \frac{\sin \theta \, d\theta}{(t \sin \theta + m(t \cos \theta))^2} \\
&\lesssim \int_1^r dt \left[\int_0^{m(t)/t} \frac{\theta \, d\theta}{m(t)^2} + \int_{m(t)/t}^\pi \frac{d\theta}{t^2 \theta} \right] \\
&\lesssim O(1) + \int_1^r \log(t/m(t)) \frac{dt}{t^2} \\
&\lesssim O(1) + \int_1^r \log \frac{1}{m(t)} \frac{dt}{t^2} \\
&= O(1),
\end{aligned}$$

completing the proof of part (a). \square

3.3 Proof of Theorem 1(b)

To prove part (b), we use the elliptic modular function λ . We will rely only on its most basic properties, which can be found in [?, Chapter 7, Sections 3.4, 3.5] as well as in [?, Section 23]. It is analytic in \mathbb{H} , 2-periodic, does not attain the values 0 and 1, and satisfies $\lambda(\tau) \rightarrow 0$ as $\text{Im } \tau \rightarrow \infty$.

Lemma 6. *There exists a positive numerical constant c such that, for all sufficiently small $y > 0$, we have*

$$\int_0^2 \log^+ |\lambda(x + iy)| \, dx \geq c \log(1/y).$$

Proof of Lemma ??: First, we move to the unit disk, letting $q = e^{\pi i \tau}$, $\tau = x + iy \in \mathbb{H}$, and $\Lambda(q) = \lambda(\tau)$, $\Lambda(0) = 0$. Then, letting $r = |q| = e^{-\pi y}$, we have

$$\int_0^2 \log^+ |\lambda(x + iy)| \, dx = \frac{1}{\pi} \int_{-\pi}^\pi \log^+ |\Lambda(re^{i\theta})| \, d\theta.$$

Hence, for any $a \in \mathbb{C} \setminus \{0, 1\}$,

$$\begin{aligned}
\int_0^2 \log^+ |\lambda(x + iy)| \, dx &\geq \frac{1}{\pi} \int_{-\pi}^\pi \log^+ |\Lambda(re^{i\theta}) - a| \, d\theta - C(a) \\
&\geq 2 \sum_{|q_k| \leq r} \log \frac{r}{|q_k|} - C_1(a) \quad (\text{by the Jensen formula}),
\end{aligned}$$

where $\{q_k\}$ denotes the set of a -points of Λ (counted with multiplicities), so, $|q_k| = e^{-\pi \text{Im } \tau_k}$, where $\{\tau_k\}$ is the set of a -points λ in $\{-1 \leq \text{Re } \tau < 1\}$. Then $\log(r/|q_k|) = \pi(\text{Im } \tau_k - y)$,

and it will suffice to show that

$$\sum_{\operatorname{Im} \tau_k \geq 2y} \operatorname{Im} \tau_k \gtrsim \log \frac{1}{y}, \quad y \downarrow 0. \quad (5)$$

It remains to make the necessary counting.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half-plane \mathbb{H} ,

$$(M, w) \mapsto Mw = \frac{\alpha w + \beta}{\gamma w + \delta}, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad w \in \mathbb{H}.$$

The modular function λ is invariant under the subgroup $\Gamma(2)$ of $\mathrm{SL}_2(\mathbb{Z})$, which consists of integer-valued matrices M satisfying $\alpha\delta - \beta\gamma = 1$, and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}.$$

Recall that $\Gamma(2)$ is a subgroup of index 6 in $\mathrm{SL}_2(\mathbb{Z})$, and that, for $\varphi \in \mathrm{SL}_2(\mathbb{Z})$, $\lambda \circ \varphi$ can take only one of the following six values

$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \lambda, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda}{1 - \lambda}, \quad \frac{\lambda - 1}{\lambda}.$$

Fix $w = i$ and consider its orbit $G(w) = \{\varphi w : \varphi \in \mathrm{SL}_2(\mathbb{Z})\}$. For any $z \in G(w)$ we have that $\lambda(z)$ is equal to one of six numbers

$$a, \quad \frac{1}{a}, \quad 1 - a, \quad \frac{1}{1 - a}, \quad \frac{a}{1 - a}, \quad \frac{a - 1}{a},$$

where $a = \lambda(w)$. If we are able to show that

$$\sum_{\substack{z \in G(w): \\ |\operatorname{Re} z| < 1, \operatorname{Im} z > 2y}} \operatorname{Im} z \gtrsim \log \frac{1}{y}$$

then by the pigeonhole principle the same would hold for at least one of six values above, thus giving us the desired claim (note that, potentially, which exact value we need will depend on y but as long as the total set of possible values is finite and fixed this is not a problem).

For a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $w = i$, we have

$$Mw = \frac{\alpha i + \beta}{\gamma i + \delta} = \frac{\beta\delta + \alpha\gamma}{\gamma^2 + \delta^2} + \frac{i}{\gamma^2 + \delta^2}. \quad (6)$$

It might happen that $M_1 w = M_2 w$ for distinct matrices M_1 and M_2 . However, in this case $M_2^{-1} M_1 w = w$ and this can not happen too often. Specifically, $Mw = w$ has exactly four solutions

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which can be easily seen from (??): we must have $\gamma^2 + \delta^2 = 1$, thus $(\gamma, \delta) = (\pm 1, 0)$ or $(\gamma, \delta) = (0, \pm 1)$ and in all four cases we can uniquely determine α, β from $\alpha\delta - \beta\gamma = 1$ and $\alpha\gamma + \beta\delta = 0$.

Thus, if, instead of summing over the orbit $G(w)$, we sum over all matrices in $\mathrm{SL}_2(\mathbb{Z})$ then we will count each point exactly four times, which does not affect the final conclusion of it being $\gtrsim \log \frac{1}{y}$. So, it remains to show that

$$\sum \frac{1}{\gamma^2 + \delta^2} \gtrsim \log \frac{1}{y},$$

where the sum is taken over all integers $\alpha, \beta, \gamma, \delta$ such that

$$\alpha\delta - \beta\gamma = 1, \quad |\beta\delta + \alpha\gamma| < \gamma^2 + \delta^2, \quad \gamma^2 + \delta^2 \leq \frac{1}{2y}.$$

Let us fix coprime numbers γ, δ . By Euclid's algorithm, there exists a pair α, β such that $\alpha\delta - \beta\gamma = 1$. All other pairs are obtained by adding or subtracting γ to α and δ to β , in which case $\beta\delta + \alpha\gamma$ will change by $\gamma^2 + \delta^2$ (this corresponds to shifting Mw by 1 to the left or to the right). Thus, for each such pair γ, δ , we can find at least one pair (α, β) such that $|\beta\delta + \alpha\gamma| < \gamma^2 + \delta^2$. Hence, it suffices to show that

$$\sum_{\substack{\gamma^2 + \delta^2 \leq 1/(2y), \\ (\gamma, \delta) = 1}} \frac{1}{\gamma^2 + \delta^2} \gtrsim \log \frac{1}{y}.$$

Finally, for a large number X , the number of coprime pairs (γ, δ) such that $X \leq \gamma^2 + \delta^2 \leq 2X$ is of order X (because the number of pairs which are coprime has a positive density, see, for instance, [?, Theorem 332]). Splitting the sum we are estimating into dyadic blocks, we see that the sum is of order $\log(1/y)$, as required. \square

Proving Theorem 1(b), we assume that the function m is tame. Otherwise, we replace m by its tame majorant m^* with divergent logarithmic integral, which exists by Lemma ???. Then $\mathbb{H}(-m^*) \supseteq \mathbb{H}(-m)$; so, if a meromorphic function F omits three distinct values in $\mathbb{H}(-m^*)$, *a fortiori* it omits the same values in $\mathbb{H}(-m)$.

As in the proof of Theorem 1(a), let $w: \mathbb{H} \rightarrow \mathbb{H}(-m)$ be the Riemann map such that $w(iy)$ is purely imaginary and $w(iy) \rightarrow \infty$ as $y \rightarrow \infty$. Set $W = w^{-1}$ and $\Gamma = W(\mathbb{R})$, that is, $\mathrm{Im} w(z) = 0$ for $z \in \Gamma$. Clearly, Γ is a smooth curve, symmetric with respect to the imaginary axis. Recall that, due to tameness of m , by Lemma ???, the mapping W is continuously differentiable up to the boundary.

Claim 7. *Let $W = U + iV$. Then, for any $y \geq 0$, the function $x \mapsto V(x, y)$ is non-increasing on $\mathbb{R}_{\geq 0}$.*

Proof. Consider the partial derivative $V_x(x, y)$. It is a bounded harmonic function in $\mathbb{H}(-m)$ (boundedness follows from Lemma ???). By the symmetry of the map W , we have

$V(-x, y) = V(x, y)$, and therefore the function V_x vanishes on the imaginary axis. The function V is positive in $\mathbb{H}(-m)$ and vanishes on $\partial\mathbb{H}(-m)$. Then its gradient ∇V points into $\mathbb{H}(-m)$. Since m is non-increasing, this implies that $V_x(x, -m(x)) \leq 0$ for $x \geq 0$ (and similarly, ≥ 0 for $x \leq 0$). Thus, by the Phragmén–Lindelöf principle, $V_x(x, y) \leq 0$ in $\mathbb{H}(-m) \cap \{\operatorname{Re} z > 0\}$. \square

By Claim ??, the curve Γ is a graph of an even smooth function $n: \mathbb{R} \rightarrow \mathbb{R}_{>0}$, non-increasing on $\mathbb{R}_{\geq 0}$: $\Gamma = \{x + in(x): x \in \mathbb{R}\}$.

Claim 8. For $x \in \mathbb{R}$,

$$n(x) \leq Cm(x/C) \tag{7}$$

with $C = \|W'\|_\infty = \sup\{|W'(\zeta)|: \zeta \in \overline{\mathbb{H}}(-m)\}$.

Proof. Since both functions m and n are even, it suffices to check (??) only for $x \geq 0$. Let, as above, $W = U + iV$. Then $V(t) = n(U(t))$, $t \in \mathbb{R}$. Since

$$V(t) = \operatorname{Im} W(t) = \operatorname{Im} [W(t) - W(t - im(t))],$$

we have

$$V(t) \leq |W(t) - W(t - im(t))| \leq Cm(t).$$

Similarly,

$$U(t) = \operatorname{Re} W(t) = \operatorname{Re} [W(t) - W(0)],$$

whence, for $t \geq 0$, $U(t) \leq |W(t) - W(0)| \leq Ct$, and therefore, $m(t) \leq m(U(t)/C)$. Combining both estimates, we obtain

$$n(U(t)) = V(t) \leq Cm(t) \leq Cm(U(t)/C).$$

Letting $x = U(t)$, we get the claim. \square

Now we are ready to prove Theorem 1(b). Assume that

$$\int_1^\infty \frac{\log^- m(t)}{t^2} dt = +\infty,$$

that m is tame, and let $F = \lambda \circ W$. The function F is analytic in $\mathbb{H}(-m)$ and does not take the values 0 and 1 there. We aim to show that F is of unbounded type in \mathbb{H} . Suppose that it is not true, and F has a bounded type in \mathbb{H} . Then

$$\int_{\mathbb{R}} \frac{\log^+ |F(t)|}{t^2 + 1} dt < \infty. \tag{8}$$

Moreover, since, for a given x , $\lambda(x + iy)$ tends to 0 as $y \rightarrow \infty$, the function F belongs to the Smirnov class in \mathbb{H} , that is, the subharmonic function $\log^+ |F|$ is majorized by its Poisson integral

$$\begin{aligned} \log^+ |F(\zeta)| &\leq \frac{\operatorname{Im} \zeta}{\pi} \int_{\mathbb{R}} \frac{\log^+ |F(t)|}{|t - \zeta|^2} dt \\ &= \int_{\mathbb{R}} \log^+ |F(t)| \omega_{\mathbb{H}}(dt, \zeta) < \infty, \quad \zeta \in \mathbb{H}. \end{aligned} \quad (9)$$

Here and elsewhere, $\omega_{\mathbb{G}}(dz, \zeta)$ denotes the harmonic measure of the domain \mathbb{G} evaluated at $\zeta \in \mathbb{G}$. Letting $\zeta = w(z)$ in (??), we get

$$\log^+ |\lambda(z)| \leq \int_{\Gamma} \log^+ |\lambda(z')| \omega_{\mathbb{H}(n)}(dz', z) < \infty, \quad z \in \mathbb{H}(n), \quad (10)$$

where, as above, $\Gamma = \{x + in(x) : x \in \mathbb{R}\} = \partial\mathbb{H}(n)$. By Claim ??,

$$\int_1^{\infty} \frac{\log^- n(t)}{t^2} dt = +\infty.$$

Let n^* be a tame majorant of n with divergent logarithmic integral which exists by Lemma ??. Then $\mathbb{H}(n^*) \subset \mathbb{H}(n)$. Let $\Gamma^* = \partial\mathbb{H}(n^*) = \{x + in^*(x) : x \in \mathbb{R}\}$, and let $w^* : \mathbb{H} \rightarrow \mathbb{H}(n^*)$ be the Riemann map with our usual convention (w^* is purely imaginary, and $w^*(i\eta) \rightarrow \infty$ as $\eta \rightarrow \infty$).

Claim 9. For $z \in \mathbb{H}(n^*)$,

$$\int_{\Gamma^*} \log^+ |\lambda(z')| \omega_{\mathbb{H}(n^*)}(dz', z) \leq \int_{\Gamma} \log^+ |\lambda(z')| \omega_{\mathbb{H}(n)}(dz', z) < \infty. \quad (11)$$

Proof. Denote by $h(z)$ the RHS of (??). This is a non-negative harmonic function in $\mathbb{H}(n)$ that majorizes $\log^+ |\lambda|$ in $\overline{\mathbb{H}}(n)$. Returning to the upper half-plane, we let $u = \log^+ |\lambda \circ w^*|$ and $v = h \circ w^*$. The non-negative functions u and v are correspondingly subharmonic and harmonic in \mathbb{H} , continuous in $\overline{\mathbb{H}}$, and $u \leq v$ in $\overline{\mathbb{H}}$. Furthermore, by the Herglotz – F. Riesz representation theorem applied to the positive harmonic function v , we have

$$v(\zeta) = c\eta + \frac{\eta}{\pi} \int_{\mathbb{R}} \frac{v(t)}{|t - \zeta|^2} dt, \quad \zeta = \xi + i\eta,$$

with $c \geq 0$. Therefore,

$$\frac{\eta}{\pi} \int_{\mathbb{R}} \frac{u(t)}{|t - \zeta|^2} dt \leq \frac{\eta}{\pi} \int_{\mathbb{R}} \frac{v(t)}{|t - \zeta|^2} dt \leq v(\zeta), \quad \zeta \in \mathbb{H},$$

which is nothing but (??) with $z = w^*(\zeta)$. □

To arrive at a contradiction, it remains to show that the integral on the LHS of (??) is infinite. Fix $z \in \mathbb{H}(n^*)$. The function n^* takes the constant value $n_k = n^*(2^k)$ on the interval $I_k = [2^k - 2^{k-3}, 2^k + 2^{k-2}]$, $k \geq 3$. Hence, the curve $\Gamma^* = \{t + in^*(t)\}$ has long horizontal intervals Γ_k over I_k of height n_k . Next, we claim that on the middle third of Γ_k the harmonic measure $\omega_{\mathbb{H}(n^*)}(dz', z)$ is uniformly in k comparable with the measure dt/t^2 .

Claim 10. *Let J_k be the middle third of I_k . Then there exists $k_0 = k_0(z)$ and constants $0 < c(z) < C(z) < \infty$ (depending on z but not on k) such that, for every $k \geq k_0$ and every Borel set $A \subset J_k$,*

$$c(z) \int_A \frac{dt}{t^2} \leq \omega_{\mathbb{H}(n^*)}(A + in_k, z) \leq C(z) \int_A \frac{dt}{t^2}$$

Proof. Indeed, let, as above, $w^* = u^* + iv^* : \mathbb{H} \rightarrow \mathbb{H}(n^*)$ be the Riemann map with our usual normalization: $u^*(i\eta) = 0$ and $v^*(i\eta) \rightarrow \infty$ as $\eta \rightarrow +\infty$. The function w^* maps an interval $T_k \subset \mathbb{R}$ onto the boundary segment $S_k = \{x + in_k : x \in J_k\}$. Then $u^*(T_k) = J_k$, and, by Lemma ??, $|T_k| \simeq |J_k|$ and $|u^*(t)| \simeq |t|$ for $t \in T_k$. By the conformal invariance of the harmonic measure,

$$\omega_{\mathbb{H}(n^*)}(E, z) = \omega_{\mathbb{H}}((w^*)^{-1}(E), \zeta), \quad z = g(\zeta).$$

Hence, for $A \subset J_n$ and $E = \{t + in_k : t \in A\}$,

$$\omega_{\mathbb{H}(n^*)}(E, z) = \frac{1}{\pi} \int_{(w^*)^{-1}(E)} \frac{\eta}{(t - \xi)^2 + \eta^2} dt, \quad \xi + i\eta = \zeta.$$

Furthermore, $|(w^*)^{-1}(E)| \simeq |A|$, $|t| \simeq 2^k$, and for k large enough, $\eta/((t - \xi)^2 + \eta^2) \simeq 1/t^2$ (with implicit constants depending on ζ , that is, on z), proving the claim. \square

At last, we return to the integral on the LHS of (??). By Claim ??, it is

$$\gtrsim \sum_{k \geq k_0} \int_{J_k} \log^+ |\lambda(t + in_k)| \frac{dt}{t^2}$$

The k -th term of this sum is

$$\gtrsim 2^{-2k} \int_{J_k} \log^+ |\lambda(t + in_k)| dt,$$

which, by Lemma ?? and 2-periodicity of λ , is

$$\gtrsim 2^{-2k} \cdot 2^k \log \frac{1}{n_k} = 2^{-k} \log \frac{1}{n^*(2^k)}.$$

Since the function n^* is non-increasing, divergence of the series

$$\sum_k 2^{-k} \log \frac{1}{n^*(2^k)} = \infty$$

follows from the divergence of the integral

$$\int_1^\infty \frac{\log^- n^*(t)}{t^2} dt = \infty.$$

This proves the divergence of the integral on the LHS of (??), and completes the proof of the theorem. \square

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Alexandre Eremenko, Mathematics Department, Purdue University, West Lafayette, IN 47907 USA, eremenko@purdue.edu

Aleksei Kulikov, University of Copenhagen, Department of Mathematical Sciences, Universitetsparken 5, 2100 Copenhagen, Denmark, ak@math.ku.dk, lyosha.kulikov@mail.ru

Mikhail Sodin, School of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel, sodin@tauex.tau.ac.il